ADVANCED MASTERS IN STRUCTURAL ANALYSIS
OF MONUMENTS AND HISTORICAL CONSTRUCTIONS

# Master's Thesis

Amr Aly Hassan Fangary

## Graphics Analysis of Gothic Vaults

UNIVERSITAT POLITÈCNICA DE CATALUNYA

Czech Technical University

Education and Culture

Erasmus Mundus

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# DECLARATION

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Year: 2010

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

I hereby declare that the MSc Consortium responsible for the Advanced Masters in Structural Analysis of Monuments and Historical Constructions is allowed to store and make available electronically the present MSc Dissertation.

University: Universitat Politècnica de Catalunya

Date: 12/07/2010

Signature: ______________________

*To Farah, Ismail and Rehab.*

*To my family.*

*To whoever seeks knowledge.*

## ACKNOWLEDGEMENTS

I would like to acknowledge the support I got from my supervisor Professor Climent Molins and for his encouragement regarding such interesting subject.

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Last but not least, I would like to sincerely thank all the friends I made in Prague and Barcelona. I am grateful to them for understanding and improving many topics.

## ABSTRACT

The gothic architecture owes its splendor to two main structural elements; the pointed arch and the vault. However, the Gothic vault itself is a repetition of several pointed arches to create an amazing system for covering the underneath compartment. This vault was subjected to a long debate regarding its structural behavior and the way it keeps its equilibrium and transmits its thrusts. Also, the rib of the gothic vault imposed many question whatever it had a decorative, structural or constructional role. Different theories were developed but none was analytically confirmed.

The aim of this study is not to find answers to the debates, but mainly to develop a practical technique of analyzing the ribbed cross vault following the theories previously developed by the researchers, to compare the results and to rationalize them. Mainly, to undertake a task, that till now, was never accomplished. And to provide a model that can be followed for the analysis of any gothic vault.

In order to achieve this task, a general historical and architectural study was carried out regarding the vaults, then the theories of the structural behavior were studied and later the different methods of the structural analysis of the vault were recognized and explained prior to the conducted analysis.

Two case studies of Catalan Gothic vaults; the main vault of the cathedral of Santa Maria del Pi and the main vault of the cathedral of Santa Maria del Mar, were chosen to be analysed following the graphical (line of thrust) limit analysis method. Both vaults were discretisized in a geometrical model and different slices were taken for each vault following the main theories of decomposition of the vaults. Then, the defined arches were analysed using a spreadsheet and for each arch the line of thrust was graphically plotted and analysed.

The obtained results were various and quite interesting. For most of the analysed arches, it was not possible to find a line of thrust within the stone voussoirs region and the line was passing either to the sound or light infill layers. It was not possible to find a realistic line of thrust for the cross ribs arches when the loads of the webs and the upper infill were applied to them. Also, the minimum and maximum values of the lateral thrusts for the arches were almost the same.

Many conclusions can be derived from this study, regarding the form and geometry of the gothic vaults, the graphical analysis method and the behavior of the vaults. However, the most important is the role of the infill layers in the equilibrium of the vault and the transmission of the lateral thrusts. Also, the cross ribs mostly don’t have a structural role. Also, the construction of this type of thin vaults required a high level of knowledge and expertise in stabilizing and keeping in the structure equilibrium.

## RESUM

L'esplendor de l'arquitectura gòtica es deu als seus dos elements principals: l'arc ogival i la volta. Amb tot, la volta gòtica sola és com una sèrie d'arcs paral·lels que permeten crear un sistema impressionant que permet cobrir grans espais. S'ha debatut llargament sobre el comportament resistent de les voltes gòtiques i de com són capaces de restar en equilibri i transmetre les empentes. A més, el nervi diagonal de les voltes gòtiques planteja nombroses qüestions sobre si es tracta d'un element decoratiu, estructural o constructiu. Tot i que s'han desenvolupant diferents teories, cap ha estat confirmada analíticament.

L'objectiu d'aquest estudi no és trobar la resposta a aquests debats, sinó desenvolupar una tècnica pràctica per analitzar voltes de gòtiques nervades seguint les teories prèviament desenvolupades pels investigadors, a fi de comparar-ne els resultats i racionalitzar-los, perquè aquesta tasca no s'ha portat a terme fins ara. A més, es vol proporcionar un model que permeti l'anàlisi de qualsevol volta gòtica.

Per aconseguir-ho, s'ha desenvolupat un estudi històric i arquitectònic general sobre les voltes, després s'han estudiat les teories sobre el seu comportament estructural i, finalment, s'expliquen els diferents mètodes d'anàlisi de les voltes abans de desenvolupar les anàlisis.

S'han escollit dos casos d'estudi de voltes gòtiques catalanes, les voltes principals de Santa maria del Pi i de Santa Maria del Mar, per analitzar-les amb el mètode de l'ànalisi límit gràfica (línia de pressions). Ambdues voltes s'han discretatzat en arcs a partir d'un model geomètric i seguint les teories principals de descomposició de voltes. Llavors, cada arc ha estat analitzat usant un full de càlcul que permet comprovar gràficament la línia de pressions.

S'han obtingut diversos resultats i tots ells interessants. Per a la majoria d'arcs analitzats, no ha estat possible trobar una línia de pressions compresa dins les dovelles ja que passava a través del reble compacte o del lleuger. Tampoc ha estat possible trobar una línia de pressions realista per als arcs dels nervis diagonals quan les càrregues de tots els arcs en què es descomposa la volta carregaven damunt seu. A més, els valors màxim i mínim de l'empenta lateral dels arcs era gairebé el mateix.

D'aquest estudi se'n deriven moltes conclusions relatives a: la forma i la geometria de les voltes gòtiques, el mètode d'anàlisi gràfica i sobre el comportament resistent de les voltes. No obstant, la més important és el paper que té el reble en l'equilibri de la volta i en la transmissió de les empentes laterals. També, que els nervis diagonals pràcticament no contribueixen a la resistència. A més, cal assenyalar que la construcció d'aquest tipus de voltes primes requeria un bon nivell d'expertesa i coneixement per fer-les estables tot mantenint l'estructura en equilibri.

**عنوان البحث: التحليل الاستاتيكي بالرسم الهندسي للاقبيه القوطيه.**

**الباحث: عمرو علي حسن فنجري**

**المشرف: أ.د./ كليمنت مولينز**

## ملخص البحث

فن العمارة القوطية يدين برونقه إلى عنصرين هيكليين رئيسيسن ، العقد المدبب والقبو .ومع ذلك ، فإن القبو القوطي نفسه هو تكرار لعدة عقود مدببة من اجل تكوين نظام متميز لتغطية الفراغ اسفله .وقد تعرض هذا القبو إلى مناقشة طويلة بشأن سلوكه الهيكلي والطريقة التي يحافظ بها على توازنه وتنقل عناصره الاحمال التي يتعرض لها. كذلك فرض العصب المكون للقبو القوطي اسئله كثيرة فيما يتعلق بالناحيه الوظيفيه ان كانت زخرفية , إنشائية أو لاحتياجات بنائيه .وقد وضعت نظريات مختلفة لكن لم يتم تأكيد أي منها بواسطه التحليل الانشائي.

والهدف من هذه الدراسة ليس العثور على إجابات لهذه المناقشات ، ولكن أساسا لتطوير تقنية عملية لتحليل القبو المضلع المتقاطع عبر النظريات التي سبق وضعها من قبل الباحثين و لمقارنة النتائج وترشيدها. فالغرض من الدراسه بصوره عامه هو إجراء المهمة التي حتى الآن لم يتم انجازها وتقديم النموذج الذي يمكن اتباعه لتحليل أي قبو قوطي.

من أجل تحقيق هذه المهمة ، تم البدء بعمل دراسة تاريخية ومعمارية عامة بشأن الاقبيه ، ثم تمت دراسة نظريات السلوك الهيكلية والأساليب المختلفة السابقه للتحليل الهيكلي للقبو والتعرف عليها و شرحها. و تم اختيار دراسه حالتين من نماذج الاقبيه القوطية الكاتالونية ، وهما القبو الرئيسي للكاتدرائية سانتا ماريا ديل بي القبو الرئيسي للكاتدرائية سانتا ماريا دل مار ، ليتم تحليلهما باستخدام التحليل الاستاتيكي بالرسم الهندسي (خط الدفع) بأسلوب تحليل الحمل الاقصي للمنشأ .كل من القبوين تم تمثيلهما في نموذج هندسي وأخذت شرائح مختلفة لكل قبو تمشيامع النظريات الرئيسية لتحليل الأقبية .ثم تم تحليل أقواس محددة باستخدام جدول بيانات حسابي وايجاد خط الاتجاه (الدفع) لكل قوس بالرسم البياني وتحليلها.

ان النتائج التي تم الحصول عليها متعدده ومثيرة للاهتمام .فبالنسبة لمعظم الأقواس التي تم تحليلها ، لم يكن من الممكن العثور على خط الدفع في منطقة احجار العقد فخط الدفع كان يمر خارجها إما الي منطقه طبقه الردم المتماسك او الي طبقه الردم الخفيف. بالتالي لم يكن من الممكن العثور على خط واقعي للدفع عبر أقواس الاعصاب التقاطعه للقبوات و ذلك عند تطبيق أحمال الاعصاب و طبقات الردم التي تعلوها .أيضا ، كانت القيم الدنيا والقصوى للقوي الافقيه للعقود تقريبا متماثله.

يمكن استخلاص الكثير من الاستنتاجات من هذه الدراسة فيما يتعلق بشكل وهندسة الاقبيه القوطية ، وطريقة تحليل الرسوم البيانية وسلوك الاقبيه .ومع ذلك ، فإن الأكثر أهمية هو الدور الذي تضطلع به طبقات الردم العليا و المحيطه في تحقيق التوازن للقبو ونقل الاحمال للاتجاهات الجانبية .أيضا فان الدراسه ترجح ان الاعصاب المتقاطعه ليس لها دورا هيكليا . خلاصه, ان بناء هذا النوع من الاقبيه الرقيقة يتطلب الي مستوى رفيع من المعرفة والخبرة في تحقيق الاستقرار وحفظ التوازن في الهيكل.

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# 1. INTRODUCTION

*"There is no actual state for any structure, whether built of masonry, steel, or any other material. There is, of course, an ideal state imagined by the designer (or by the computer making the design), and there is in practice a state here-and now, but any small disturbance- a small foundation settlement, a lurch in the wind, an earth tremor, a decay of mortar, a slip in a connexion – will cause a huge alteration to the values of internal stresses."*

The previous words of Jacques Heyman provide a deep understanding of the problem of dealing with the structures, not only the historical but also the modern. The analyst is not obliged to determine the 'actual' state of the structure; he is required to find any satisfactory equilibrium solution and this will demonstrate the stability of the structure. It is almost impossible to find the true or actual state of a building, but clothing a reasonable equilibrium pattern of forces with suitable constructional material will mean that the structure is safe not collapse under those loads.

The increase of interest in the historical architectural heritage and the need of preservation for the historical structures during the last years led to a continuous development of growing number of methods for the analysis of masonry vaults. Still there is a general lack of scientific and experimental knowledge in this field. The components of the historic structures including its material properties, its structural behaviour and there combinations are mostly ambiguous. A consequence of abandoning the old building practice or kept of the knowledge without transmit by the traditional builders.

Still a traditional limit analysis method, the graphical analysis can be favored for studying the equilibrium of a structure due to its efficiency, economy and practicality. Specially, when dealing with the gothic vaults which were a subject of debate till today regarding their structural behavior.

The aim of this study is not to find answers to the debates, but mainly to develop a practical technique of analyzing the ribbed cross vaults following the theories previously developed by the researchers, to compare the results and to rationalize it. Mainly, to undertake a task, that till now, was never accomplished. And to provide a model that can be followed for the analysis of any gothic vault.

This study will cover the historical and architectural concept of the gothic vault in the second chapter. Then, it will investigate the development of the theories governing the structural behavior and its evolution in chapter three. The different types of analyses for the vault are described in the forth chapter, and examples are also provided with the emphasis on the graphical limit analysis method. The fifth chapter is dedicated to the practical analyses of two case studies of gothic vaults; the main vault of the cathedral of Santa Maria del Pi and the main vault of the cathedral of Santa Maria del Mar. both churches are located in Barcelona, Catalunya. In this last chapter, the concepts and the followed process are precisely explained, and then the observations and the results are discussed. Finally, the last chapter is to enclose the general conclusions acquired by such an interesting experience.

# 2. THE GOTHIC VAULT

The vault is not only an architectural pattern existing in a building, but it is also a structural system that seeks global equilibrium and stability of the whole structure. Starting from the roman vault till reaching the high gothic vault, a long story of evolution and adaptation to culture and environment took place. The multiple types of vaults evolved and developed due to economic, structural, practical and aesthetic reasons. Different studies went on digging to recover all the ambiguity and mystery about the vaults, and specially the gothic vaults. Although these studies clarified many issues, they left other questions till today unsolved.

This chapter has no intendance to explain all the previous studies or to reveal the doubts that still exist. But it aims to define the main terminology used for dealing with the historical arches and vaults. It also gives a general idea about the evolution of the vaults and why the gothic vaults are defined as the state of art for such constructions. Moreover, it will discuss some theories about the construction process. This is crucial from the point of view of the researcher to have a better understanding of this structural element to be able of practically analyzing the gothic vaults and to realistically understand its behavior.

## 2.1 Definitions

### *2.1.1 Gothic architecture:*

Architectural construction was the dominant expression of the Gothic Age. Emerging in the first half of the 12th century from Romanesque antecedents, Gothic architecture continued well into the 16th century in northern Europe, long after the other arts had embraced the Renaissance. Although a vast number of secular monuments were built in the Gothic style, it was in the service of the church, the most prolific builder of the middle Ages that the new architecture evolved and attained its fullest realization. (Rudolph 2006)

The aesthetic qualities of Gothic architecture depend on a structural development: the ribbed vault. Medieval churches had solid stone vaults which were extremely heavy structures and tended to push the walls outward, which could lead to the collapse of the building. In turn, walls had to be heavy and thick enough to bear the weight of the stone vaults. Early in the 12th century, masons developed the ribbed vault, which consists of thin arches of stone, running diagonally, transversely, and longitudinally. The new vault, which was thinner, lighter, and more versatile, allowed a number of architectural developments to take place.

Although the earliest Gothic churches assumed a wide variety of forms, the creation of a series of large cathedrals in northern France, beginning in the second half of the 12$^{th}$ century, took full advantage of the new Gothic vault. The builders of the cathedrals found that, since the outward thrusts of the vaults were concentrated in the small areas at the springing of the ribs and were also deflected downward by the pointed arches, the pressure could be counteracted readily by narrow buttresses and by external arches, called flying buttresses. Consequently, the thick walls of Romanesque architecture could be largely replaced by thinner walls with glass windows, and the interiors could reach unprecedented heights. A revolution in building techniques thus occurred (Frankl 2000).

### *2.1.2 Masonry structures:*

According to (Heyman 1995) the definition of a masonry structure is “*an assemblage of stones - or bricks, or sun-dried clay - classified with certain distinct labels, as Byzantine, Romanesque, Gothic, adobe, but recognized by engineers as having a common structural action*”. Thus the masonry building maybe regarded as a collection of dry stones (or bricks etc.), some squared and well fitted, some unworked, and placed one on another to form a stable structure. Mortar may have been used to fill interstices, but this mortar will have been weak initially, and may have decayed - it cannot be assumed to add strength to the construction. Stability of the whole is assured, in fact, by the compaction under gravity of the various elements; a general state of compression exists, but only feeble tensions can be resisted.

The magnitudes of the compressive stresses in masonry are very small. The most highly loaded portions of a great cathedral have average stresses under one-tenth of the crushing strength of the material; the stresses in a masonry roof, whether it be the dome of a Renaissance church or a Gothic rib-vault, reach perhaps one-hundredth of the crushing strength.

### *2.1.3 The Arch:*

The arch, according to The Columbia Electronic Encyclopedia, is the spanning of a wall opening by means of separate units (such as bricks or stone blocks) assembled into an upward curve that maintains its shape and stability through the mutual pressure of a load and the separate pieces.

Arches are designated according to the form of the intrados as semicircular, segmental, elliptical, and pointed, etc.

A very concise definition of the different geometrical parts of the arch was done by (Sondericker 1904) in his book Static Graphics. The terms as intrados, extrados, spandrel, crown and others are defined and explained using the drawing shown in figure 1.

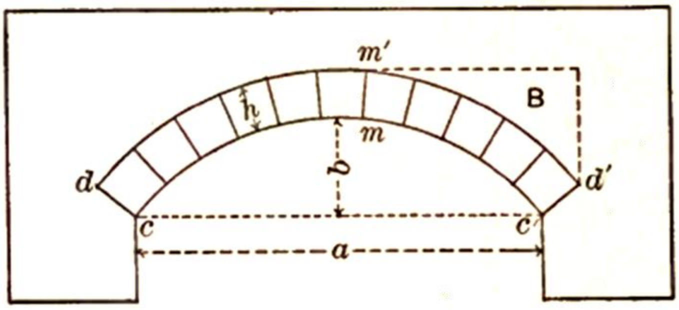



Figure 1 - definitions for the different parts of the arch (Sondericker 1904)

Where *a is the span of the arch, b is the rise of the arch, the intrados (cmc') is the inner surface of the arch ring, the extrados(dm'd') is the outer surface, the crown is the highest part of the arch, the skew-backs are the surfaces cd and c'd', the haunches are the portions of the arch ring between the crown and skew-backs, the spandrel is the space B outside the extrados and within the dotted lines, the backing is the masonry lying in space B.*

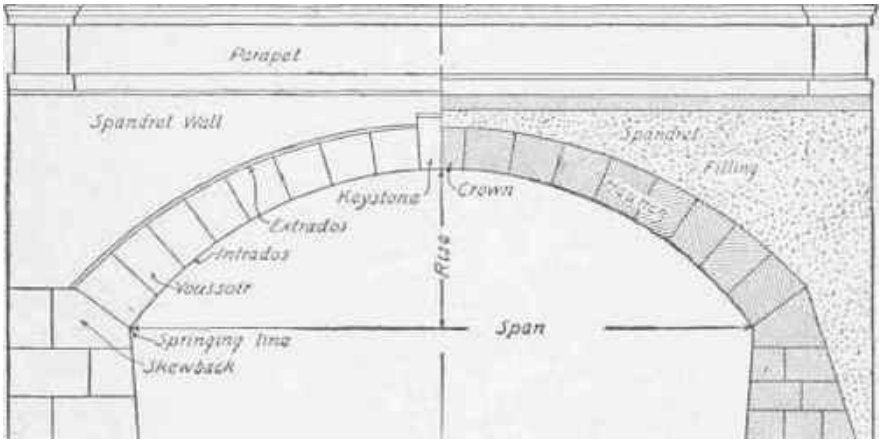



Figure 2 - terminology of the different parts of the arch.

### 2.1.4 The vault:

A vault can be defined as an arched covering over an apartment. Vaults usually take their name from the nature of the curve forming the intrados of their cross section. (Modern buildings, vol5).

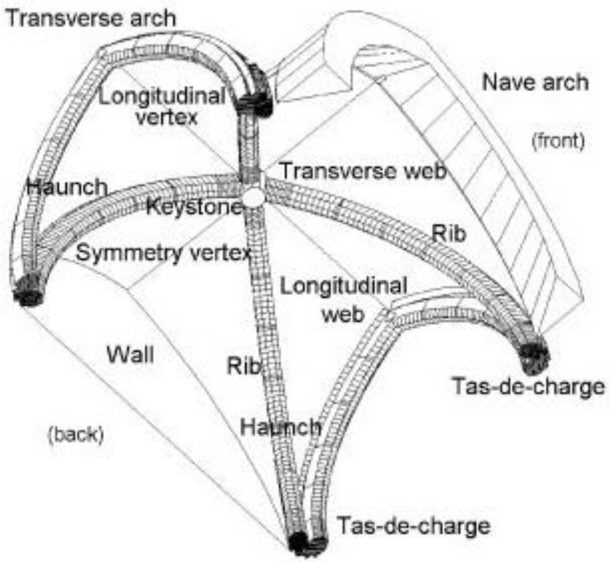



Figure 3 - Quadripartite aisle cross vaults: Geometry and definitions (Theodossopoulos 2004).

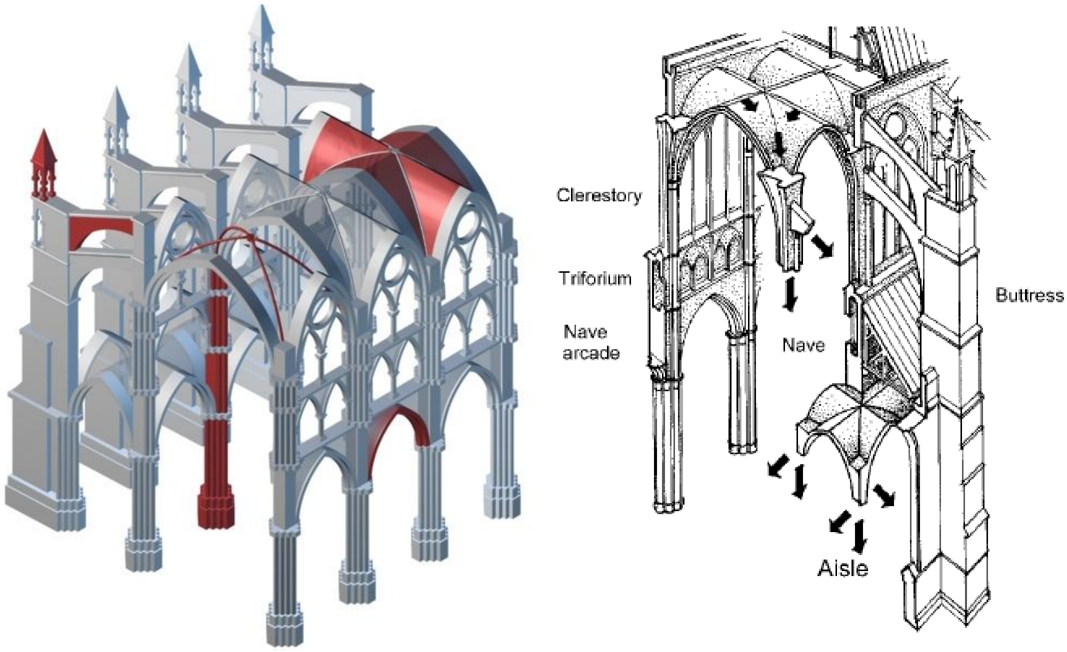



Figure 4 - Free body diagram of a cathedral (Theodossopoulos 2004).

## 2.2 Evolution of the gothic vault:

The evolution of the vault can be traced following different approaches; stylistic, aesthetical, geometrical or structural. However, it is more relevant for this study to follow the geometrical and structural one. Starting by the barrel vault, passing by groined to gothic and ending with the fan vaults, will be more explained in the following section with deeper discussion regarding the rib vault. A comparative drawing of these vaults is shown in figure 5.

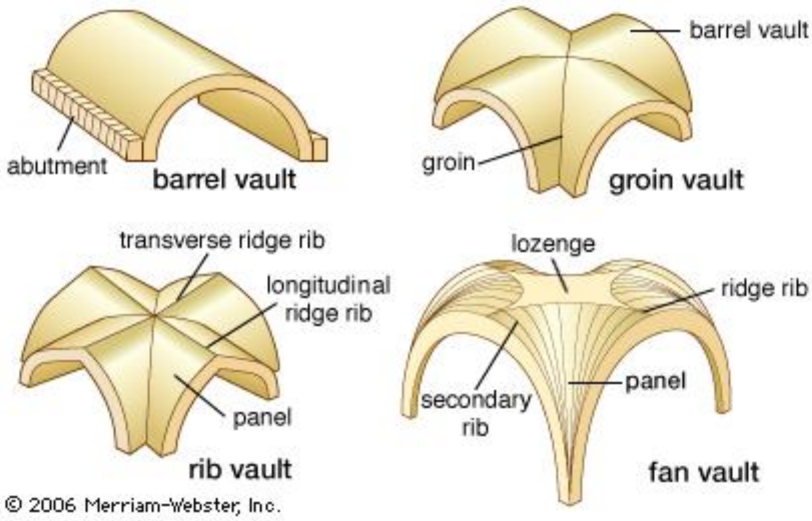



Figure 5 - Different types of vaults (Merraim-Webster)

### 2.2.1 *The barrel vault:*

A barrel vault is the simplest form of a vault consisting of a continuous surface of semicircular or pointed sections and resembles a barrel or tunnel cut lengthwise in half (Glossary of Medieval Art and Architecture). The roman builders were the first to widely use the arch system to cover their spaces. Following the arch theory, they developed the barrel vault which consists of several circular arches adjacent to each other and normally having a rectangular plan.

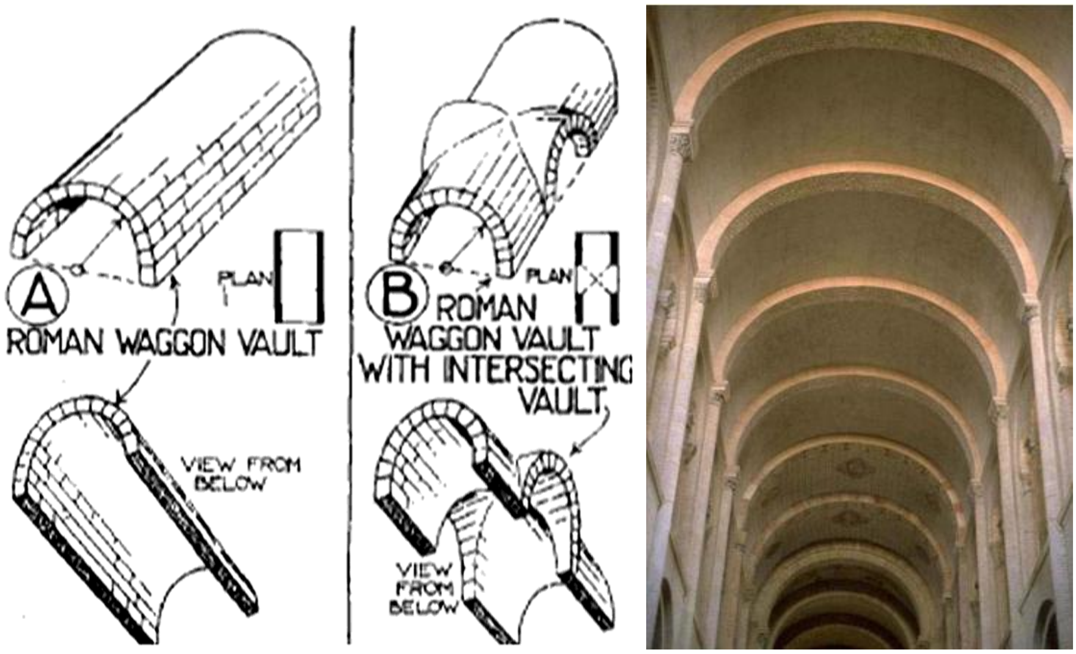



Figure 6 – The barrel vault (Fletcher 1961)

### 2.2.2 The groined vault:

The groined vault is produced by the intersection at right angles of two barrel vaults. The word groin refers to the edge between the intersecting vaults. In comparison with a barrel vault, a groin vault provides good economies of material and labor. The thrust is concentrated along the groins or arrises (the four diagonal edges formed along the points where the barrel vaults intersect), so the vault need only be abutted at its four corners.

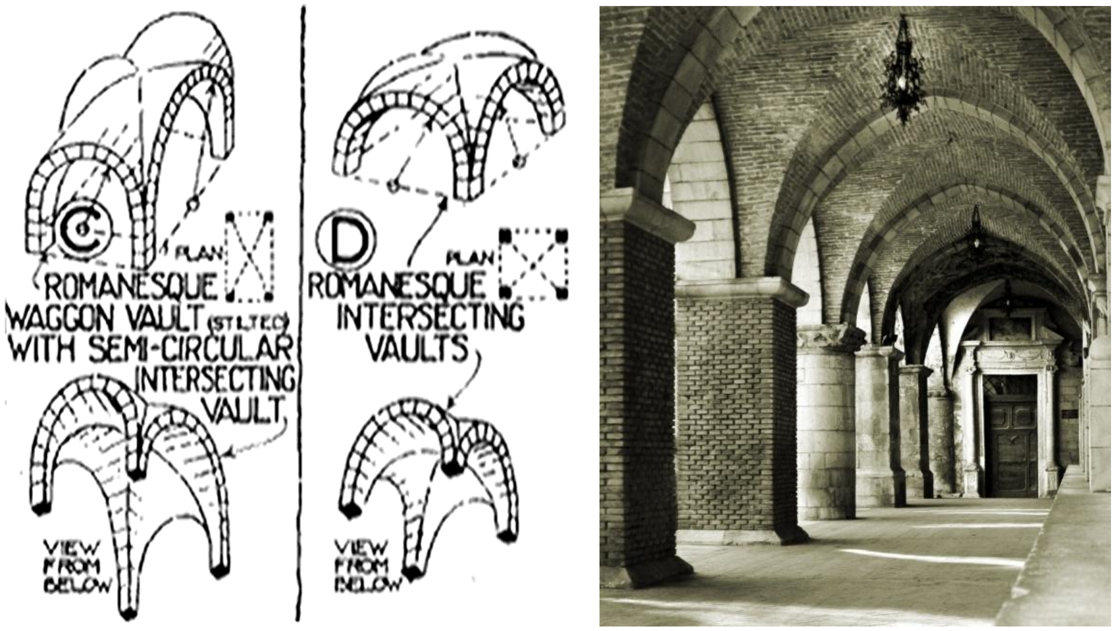
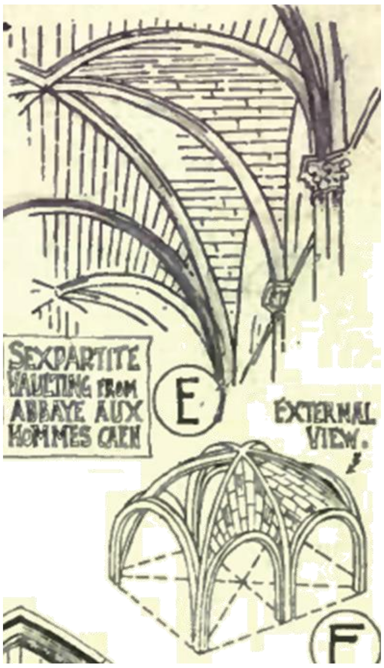



Figure 7 - example of a groined vault (Fletcher 1961).

### 2.2.3 The gothic (ribbed vault):

The gothic vault was an extension of the Romanesque system, which was evolved from that of the Romans and consisted of a framework of independent ribs, which were first constructed and which supported thin panels of stone. The difficulties of vaulting of long compartments were overcome by the introduction of the pointed arch, which was used to cover the shorter spans, while the semicircular arch was sometimes used for the diagonal ribs. The ribs became permanent centers on which the panels or "infilling" of thin stone could rest, and enabled the building to be erected all at once or in parts without disadvantage to the solidity of the edifice (Fletcher 1961).

a) 

b) 

Figure 8 - Example of the ribbed vault a) sketch (Fletcher 1961 b) Mallorca Cathedral (researcher)

According to (Porter 1911) in his book THE CONSTRUCTION OF LOMBARD AND GOTHIC VAULTS, he explains the evolution of the gothic vault and emphasizes the need of the masons to minimize the amount of used wood for the centering during the construction.

*"Rib vaults therefore were invented in Lombardy as a simple device to economize wood. They were adopted by the French builders for the same purpose. The same desire to dispense with temporary wooden substructures governed the development of architecture during the entire transitional period, and eventually lead to the birth of Gothic."*(Porter 1911)

Its introduction marked a distinct change in the design of church interiors. Perhaps most important, ribbed vaulting allowed greater fenestration of the walls than did earlier barrel and groin vaults. But there were constructional limits on how far this advantage could be carried until the flying buttress was introduced to stabilize the clerestory wall in the last quarter of the twelfth century. (Taylor & Mark 1982)

The form of the ribbed vaulting was also altered soon after the advent of the flying buttress. Prior to the year 1200, vaults sprang from a solid wall at a point below the base of the clerestory and, with few exceptions, square-planned vaults of sexpartite figuration were used in the main bays of the larger Gothic churches.

In the original construction of all High Gothic churches after 1200, there was a shift to rectangular-planned, quadripartite vaults, sprung from a point above the base of the clerestory. A causal relationship between the development of the raised High Gothic clerestory supported by flying buttresses and the shift in vault configuration can thus be accepted prima facie, yet the literature on Gothic architecture is rather vague on this point. Those explanations that have been advanced generally fall into two categories: stylistic and constructional. Implicit in all of these is the understanding that the use of sexpartite vaults arose from the introduction of alternating nave piers. Since the number of vault ribs that spring from the piers is alternately one and three for sexpartite vaulting, this system is claimed to be a more logical visual complement to alternating piers. By the same reasoning, the stylistic theories attribute the adoption of quadripartite vaulting to the introduction of uniform, non-alternating piers

Another progress occurred was the subsequent rejection of the sexpartite vault in favor of the quadripartite system. Hypothesis still fails to explain the suddenness of the change that took place. Nor do the constructional theories provide an adequate explanation. These are based on the premise that quadripartite vaults were easier to construct than sexpartite vaults; and since erecting the centering for the vaulting was one of the most complex, expensive, and dangerous operations of tall church construction, any vaulting system requiring less complicated centering would have been favored by the medieval builders.

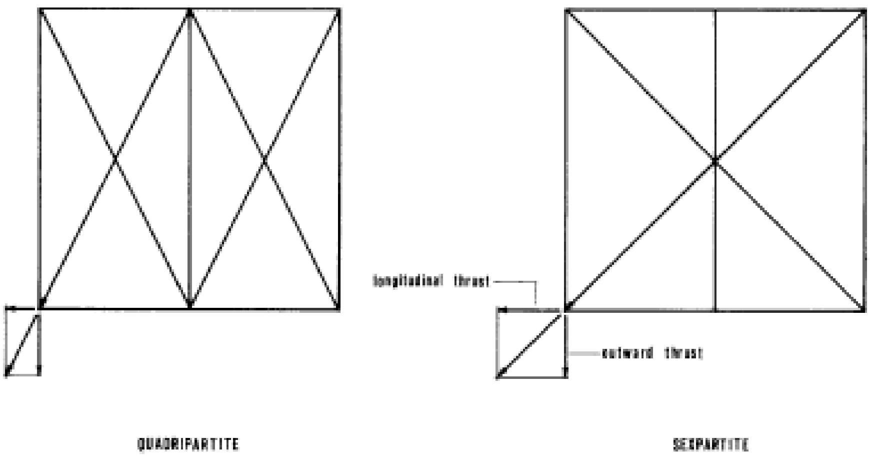



Figure 9 - Comparative plans of quadripartite and sexpartite vaulting showing longitudinal and outward thrust components (Taylor & Mark 1982)

Viollet-le-Duc maintained that the primary reason that the sexpartite plan was abandoned was because the square sexpartite bay required a diagonal rib that was much longer than the transverse rib. This raised constructional problems because the arches of the transverse ribs had to be very acute or stilted in order to attain the same heights as the keystones of the diagonal ribs The simple shape of the quadripartite vaulting, in Viollet-le- Duc's view, demanded less complex centering than did the sexpartite case (Taylor 1982).

The pointed, ribbed, four-part (quadripartite) vaults that transmit thrusts to a series of point column supports and, thus, form a vital part of the classical Gothic skeletal system was also subjected to various discussions. The rib of the vault was the main issue and what ether it was ornamental or structural. An influential illusionist with some knowledge of statics, as well as the experience of having seen a number of war-ravaged cathedrals, Pol Abraham, concluded in 1934 that the vault rib is simply an ornamental reinforcement of the angles of the vault and the fact that it projects from the vault is of no structural significance whatsoever. Abraham's view became so widely accepted that, in 1951 (Mark 1973).

### 2.2.4 The fan vault:

A fan vault is a form of vault used in the Perpendicular Gothic style, in which the ribs are all of the same curve and spaced equidistantly, in a manner resembling a fan. The initiation and propagation of this design element is strongly associated with England.

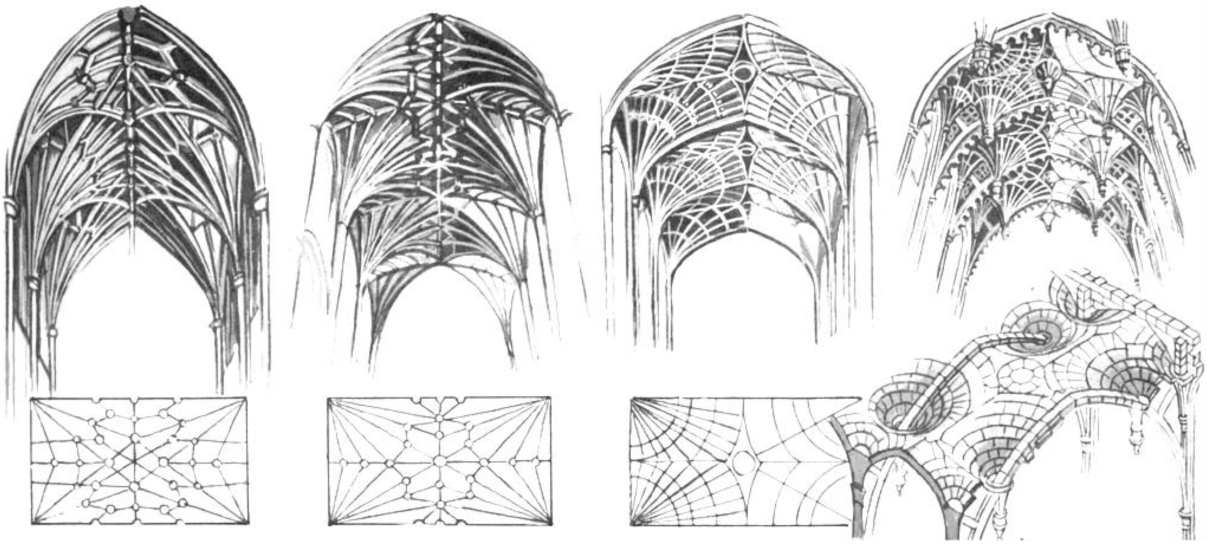


Figure 10 - different examples of fan vaults (Mansbridge 1999).

Figure 11 - Fan vaults of the Cloister at Gloucester cathedral, England, c. 1395

## 2.3 Geometry of the gothic vault:

According to Kulig, Gothic builders considered themselves “masters of geometry”. It is difficult to define the level of Gothic architects’ knowledge as the written sources and residual texts are very scarce. The building rules of craftsmen were protected by strict regulations on protection of information and were being made available in the form of verbal message exchanged between the members of a given guild only. Scraps of this knowledge were deduced from some sketchbooks and texts originating from late Gothic when some relaxation of the above-mentioned regulations took place (Kulig 2008).

Viollet-le-Duc also related an extremely simple geometric formula to determine the thickness of the columns (Figure 12) in relation to the thrust to be carried (Antonioni et. al 2007). It basically consists of dividing a half-circle into three equal parts: the direction of the thrust will be given with sufficient accuracy by the direction of two lateral segments. As can be readily grasped, pointed arches will be matched by more slender shoulders.

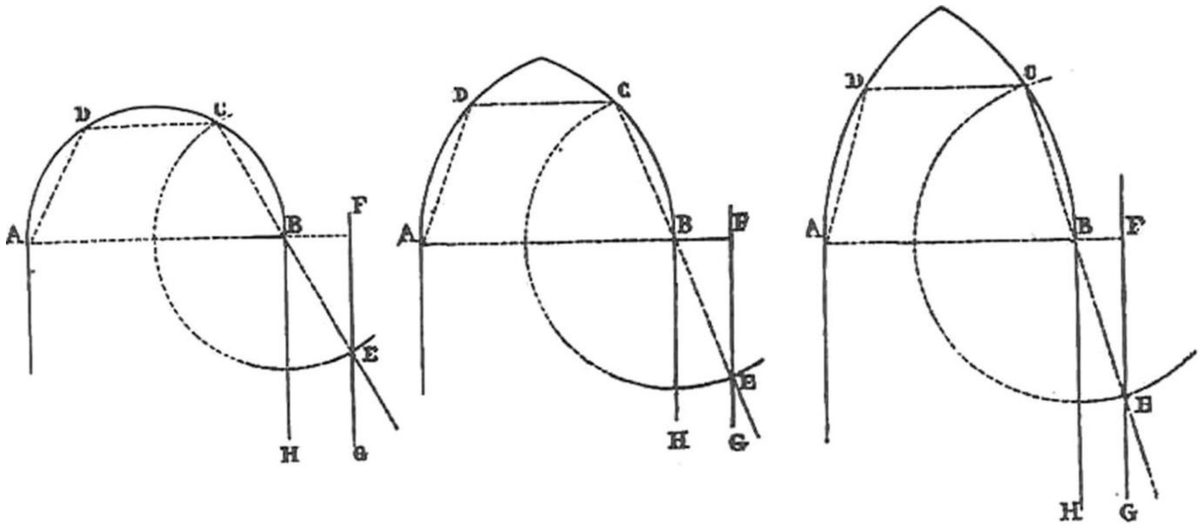



Figure 12 - Determination of the strength needed for the masonry shoulder to withstand the thrust of an arch of vault (Viollet-le-Duc after Antonioni et. al 2007).

Huerta explains these old traditional rules for the design of masonry vaults and buttresses (Huerta 2001), that they define certain proportions between the structural elements. Geometrically, the thickness of the buttress is a certain fraction of the vault's span.

## 2.4 The used material:

The materials used for Medieval masonry constructions can mainly be described as heterogeneous, irregular, anisotropic and having inconsistent (non uniform) mechanical properties. However, they have a great strength in compression. Stones, bricks or rubble are used with mortar or with dry joints. It is to be noted that the mortar, when used, is very weak in tension thus the interaction between the different elements relies on the compression forces and the force of friction. Mostly, external layers built with regular ashlar masonry, enclose irregular internal layers of infill or rubble. This fact is implicit due to many historical investigations such as those conducted by (Violet-le-Duc 1858).

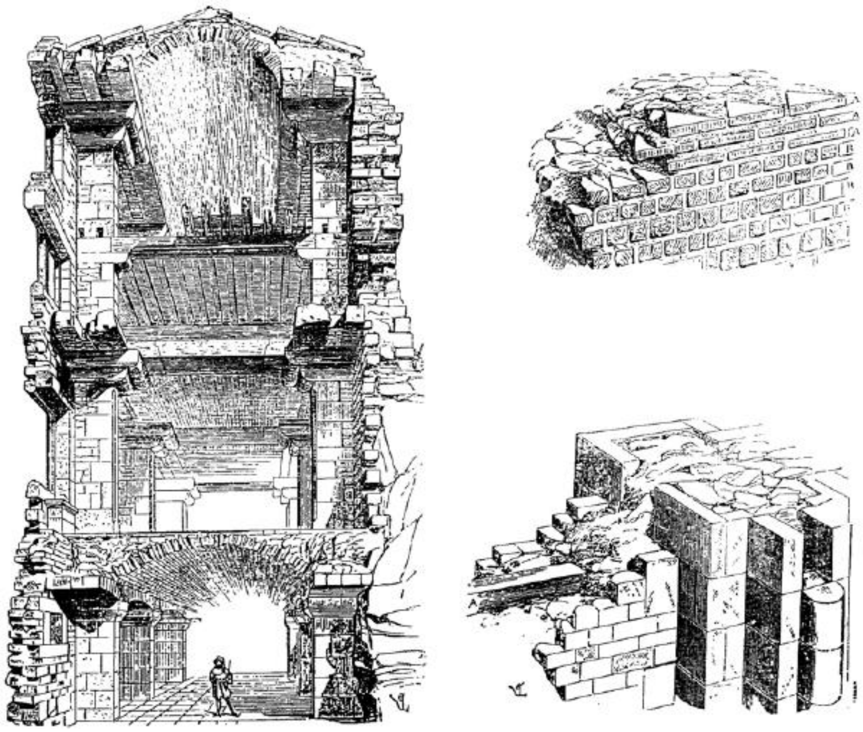


Figure 13 - Constructive section through a medieval building. Details of the construction of roman and medieval walls (Viollet-le-Duc 1858)

As for the vaults, similar to arches, they were constructed from monolithic mass contrary to the buttresses which were multilithic mass. This monolithic mass usually embodied stone blocks, usually carved and precisely cut.

## 2.5 Construction of the gothic vault:

Different theories were developed about the construction of the gothic vault dealing with the process, the centering and sometimes the order of construction. Opinions range from those of the so-called "rationalists" (or "functionalists") who view all of the elements of the Gothic buildings as examples of optimum design, to those of the anti-rationalist "illusionists" who are appalled by the idea that such great beauty could arise solely from technical considerations.(Mark 1993)

According to (Acland 1972) the geometric skills of the architects by the twentieth century allowed them to layout the groins of a cross vault on a square compartment with reasonable accuracy, but not for the vaults constructed over oblong compartments. In this case, the masons encountered difficulty in achieving a geometrically true line of intersection due to their use of small split rubble stones for the panels of their vaults. Acland puts forward that the solution was to first build a masonry rib as a support and as a cover for this erratic line of intersection. This can give an explanation for the emerging use of the rib. (Acland 1972)

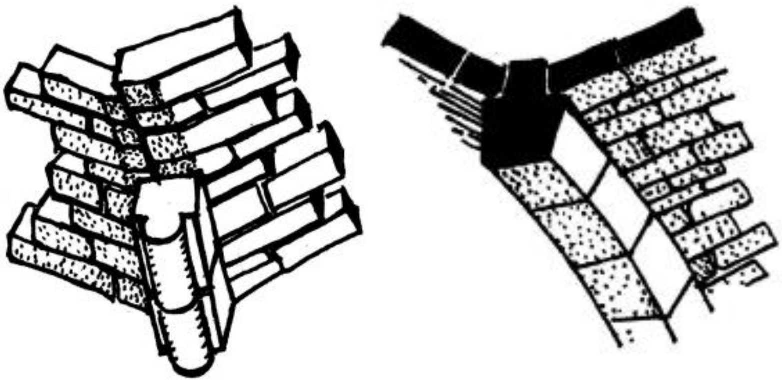


Figure 14- the rib as a solution for to conceal the erratic intersections of the vault courses. (Acland 1972)

Viollet-le-Duc and Choisy (Porter 1911) have both recognized that Gothic vaults were erected practically without the use of other centering than a light frame for the ribs and a cerce for the upper courses. The ribs of a vault were always self-supporting arches, even when broken or curved in plan.

Acland also approved this fact and accentuate it; "*On light centering frames, masons built the arches and ribs of cut stone, and then laid over them the panels of vaulting*". He explained further that Initially the centerings probably were full semicircular frames or lagging units. They were set on wedged supports which could be knocked out or decentered after the masonry had set. For longer spans it was convenient to use four frames converging on a decentering wedge. This would clear the working platform of the forest of poles necessary earlier and thus the frames could be reused for the next bay of vaulting. (Figure 15) shows the system of scaffolding for the construction of the pointed gothic vaults.

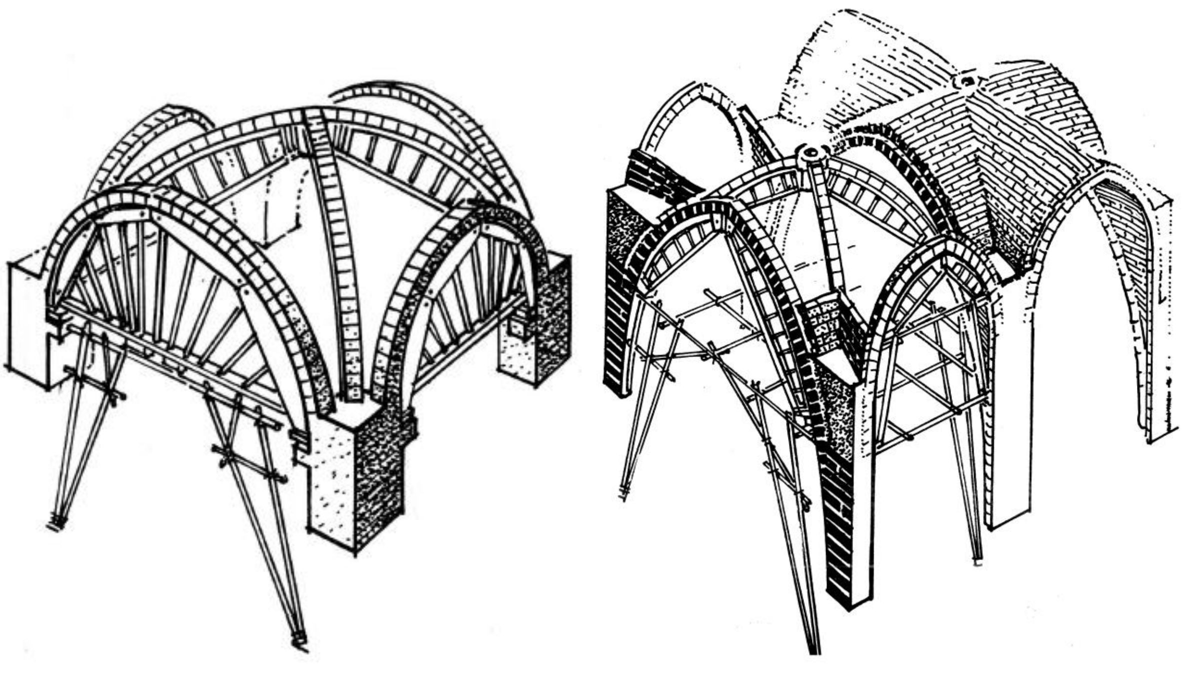


Figure 15 - Pointed arches of the vault allowed the economical use of light scaffolding. (Acland 1972)

It is to be noted that Acland confirms that the ribs were used as masonry centering frames guiding and supporting the curved webs of stone. This construction technique allowed breaking the vault into a complex of facets, which freed the designer from the need to use cumbersome and expensive continuous centering so the designer could proceed in a rational step-by-step building operation.

Also, (Theodossopoulos 2008) pointed that the crucial interlocking at the groin was largely facilitated by the ribs during the construction of the English ribbed vault. He clarified that the rubble masonry was often used to bypass the complex stereotomy. Overall, the ribs served as permanent scaffolding for the formwork of the shells, improving the quality of the intersection and alignment along the groin (figure 16).

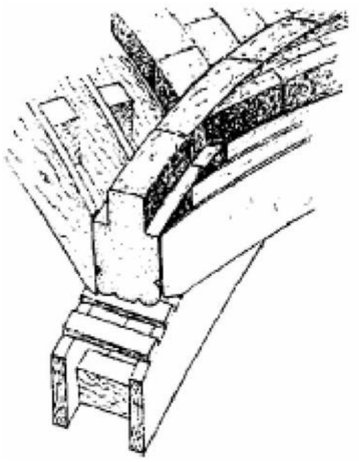


Figure 16 - Construction of the webs around the rebate of the ribs (Fitchen1961 after Theodossopoulos 2008)

Moreover, (Acland 1972) acknowledged the use of massive fill introduced at the haunching of the vault together with the heavy walls to be sufficient to restrain the considerable thrust generated by this comparatively thick structure. Then, by comparing the transversal arch to the diagonal arch ribs, he concluded that the rib was mainly for construction and not for structural reasons;

"*The heavy transverse arch rings, rectilinear in section, clearly developed from the stiffening arch rings of the barrel vault. The diagonals, lighter in section, evidently had a constructional, rather than structural, significance: as a moulding covering the awkward zig-zag intersection of vaulting panels and as built-in centering devices ensuring a true curve at the groin. They evolved from the technique of erection, the modes of building, rather than from a static analysis of the play of forces within the vaulting shell.* ." (Acland 1972)

Finally, Acland clarified that geometry of the pointed arch was brought to Europe by the returning crusaders from the Muslim lands. The pointed arch became the touchstone which freed the Christian church from the dominance of the heavy wall and the classic weight of the round arch.

## 2.6 Conclusions:

- The evolution of the intersected vault was mainly related to economical aspects; such as the use of materials, centering and scaffoldings. But also during this journey, the masons developed aesthetical aspects and geometrical skills.
- Till now, it is not possible to define the structural theories followed by the gothic masons and how the stability of the structure was achieved, but it can be concluded that they mostly learned from their success more from their failure.
- The design of the gothic structures followed imperial rules of geometry which were transmitted through the practicing cults and masons, also it seems that the rules were kept as kind of secrets and were forbidden to the public.
- The introduction of the pointed arch geometry to the gothic vault did not require the same heavy weights needed for stability which was not the case for the circular arch. Also, it freed the designers to make architectural innovations such as the more opened spaces, the light and windows decorated walls. Also there were structural innovations such as the freedom and flexibility to concentrate the loads and thrusts to specific points for instance the slender piers and buttresses.
- The ribs of the gothic vault, from a historical architectural point of view, were mostly decorative and constructional features due to the fact that they were not needed for structural purposes.

# 3. THE STRUCTURAL BEHAVIOR OF THE GOTHIC VAULT

During the medieval time of the gothic vault construction, the masons' knowledge was only related to the geometry and proportions. Following a try an error method and by vocally transferring their knowledge to their apprentices, they didn't have a clear and scientific structural understanding of the vault. It was only in the renaissance period that emerged these types of studies. And since that time, a continuous debate existed regarding the understanding of the structural behavior of the Gothic vault. Several theories were invented in order to reveal this ambiguity. Not only some of the theories were contradicting but also neither the physicists nor the historians did agree among themselves on a clear and approved theory for the supporting role of the ribs of the vault. Can the vault be considered as a simple arrangement of arches or it is more complicated. And with the evolution of computer sciences and Modeling software, the debate was still continuous.

However, understanding the structural behavior of the gothic vault implies the global understanding of the masonry arch behavior and its stability. Also, the safety of a masonry structure is a matter of geometry. A safe state of equilibrium is achieved through a correct geometry. Both historically and theoretically the "equilibrium approach" is seen by many researchers as the best approach to the analysis and design of masonry structures.

This chapter is dedicated to the theories governing the behavior of the vault, which require the understanding of the voussoir arch and the relationship between the arch and the vault. The different theories proposed by the researchers to understand the structural behavior will be explained. Later, the stability of the vault is to be discussed and finally the cracks development and their possible impact on the collapse of the vault are to be clarified.

## 3.1 Robert Hooke's law:

In 1675 the English scientist Robert Hooke (1635-1703), discovered the shape of an arch although he couldn't describe the shape mathematically. He wrote his discovery as an anagram in Latin, which descrambled reads "*ut pendet continuum exile, sic stabit contiguum rigidum in- versum",* and translates to "as hangs the flexible line, so but inverted will stand the rigid arch". Both the arch and the hanging chain must be in equilibrium, and the forces are simply reversed; since the chain can only carry tension, the arch therefore acts under compression. (Schenk 2009)

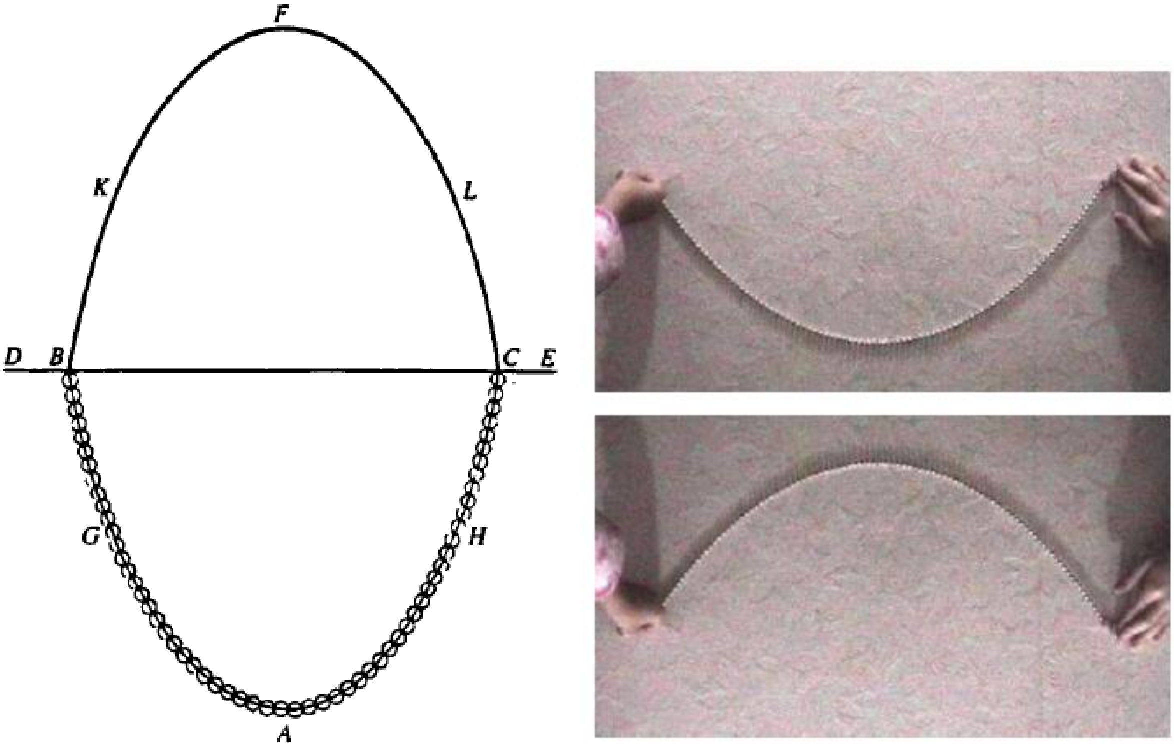



Figure 17 -The mechanics of the perfect arch in compression are the same as those of the hanging chain in tension (Poleni 1748)

Hooke, as stated by (Heyman 2000), did not put the mathematics to his statement (this was done by Gregory, 1697), but the powerful idea can be simply illustrated with the famous figure of (Poleni 1748) (figure 17); the lower part shows a string loaded with uniform weights being in tension, and in the upper reflected figure, the string itself still having the same form, shows the ideal shape of an arch to carry the same weights in compression.

This equivalence of Hooke's law was well understood in the seventeenth century, although the mathematics was not easy to solve or either proved. However, for the early masons, the main principle for the design of the arch is that it must be kept in compression and this was followed by the followers.

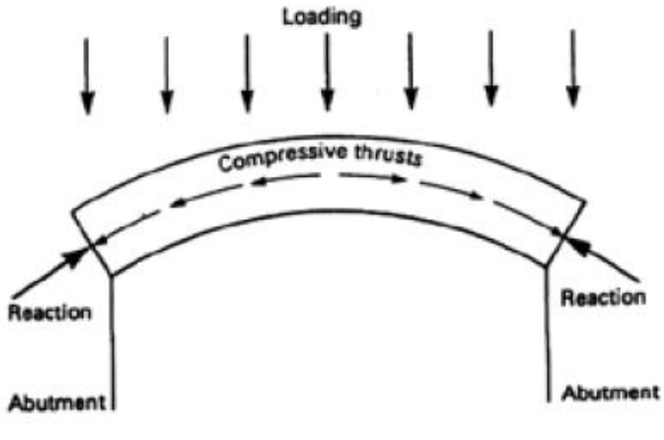



Figure 18 - The arch should be kept under compression

Therefore, by designing the shapes of an arched structure, such that the external loads are carried as compressive forces, we can create very thin domes and vaults. This type of empirical knowledge to the master builders of the Middle Ages, has resulted in marvelous structures (Schenk 2008)

## 3.2 The voussoir arch:

A voussoir arch, which consists of identical wedge-shaped blocks of stone being assembled without mortar on temporary formwork (centering) stands up after the removal of the centering. Robert Hooke, in his book on helioscopes, explained that the arch can carry its own weight.

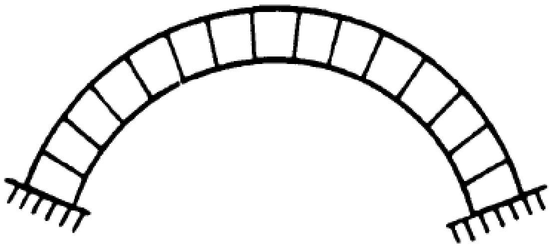


Figure 19 - The simplest masonry structure - the voussoir arch (Heyman 2000).

As mentioned by (Heyman 2000) after Gregory that if an arch of some form other than the perfect ‘catenary’ were stable, it would be because 'in its thickness some catenaria is included'. The form of the hanging string maybe modified by shortening or lengthening the string (or by changing the distance between the points of support), and it seems clear that a catenary of the general shape can be found to fit between the circular extrados and intrados. Heyman clarifies that a full semicircular arch must have a thickness of about 10% of its radius if it is to contain such an inverted catenary. And, this required thickness reduces very rapidly if the arch embraces less than a semicircle.

The inverted catenary represents the thrust line for the arch; it indicates the path along which the compressive forces are transmitted from voussoir to voussoir and so to the abutments. There is a wide range of possible positions for the line of thrust to exist within the arch and any one of these positions can confirm the stability of the arch. Heyman stresses on the importance of the seventeenth-century statements of Hooke/Gregory to understand the phenomena; “*if a position of the catenary can be found which lies within the boundaries of the masonry, then the structure is satisfactory*.” (Heyman 2000)

## 3.3 The Stability:

Charles-Agustin COULOMB (1736-1806) proposed in 1773 the first general and accurate theory on the stability of masonry arches. The basic assumptions are; “*Sliding between voussoirs is unlikely due to the existing frictional forces. Collapse will be caused by the rotation between parts due to the appearance of a number of hinges. The location of the hinges is a priori unknown but can be determined by the method of maxima and minima*”

Heyman (2000) generalized the ideas of Hooke and Coulomb, and concluded that the design of masonry structures, no matter how complex, is seen to be one of assigning the correct geometry to the various elements. The 'lines of thrust' represent the loads which the structure is required to carry. These thrusts must then be 'clothed' in masonry of sufficient thickness to contain them. Gothic architects used their medieval rules of construction (and Vitruvian rules) which were concerned with geometry and with correct proportion. They also understood the necessity of buttresses or abutments so they designed it by numerical rules of proportion (Heyman 2000).

Heyman explained the behavior of the arch when the centering is removed after construction. He pointed that the arch will at once start to thrust horizontally at its abutments; the abutments will, inevitably, give way by small unknown amounts. There will then be a small geometrical mismatch; if the arch does not fall, it must somehow accommodate itself to the slightly increased span. This accommodation is made by cracking, as shown (greatly exaggerated) in Figure 20. The voussoirs cannot slip, and they cannot deform within themselves, but they can turn about points of contact with each other either on the extrados or on the intrados; the resulting cracks, which maybe hairline, or even closed in practice by the real elasticity of the stone, may be idealized as hinges.

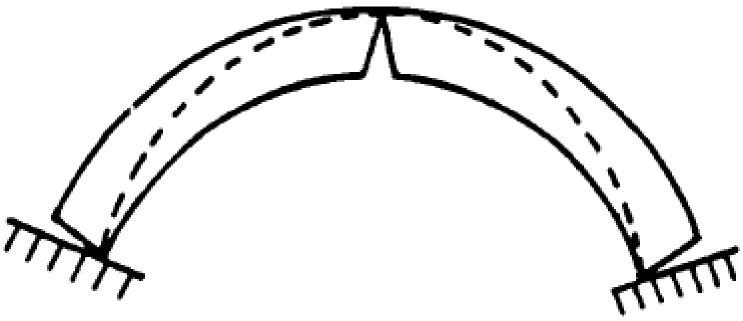


Figure 20  A structure will never, in practice, fit perfectly to its foundations. A masonry arch will crack If the abutments spread slightly, but will nevertheless be a perfectly satisfactory structure (Heyman 2000)

Such cracks will always occur in a stone structure. For the arch the abutments will spread, leading to a simple pattern of cracks. For a complex masonry structure, settlement and spreading of the foundations, and drying and shrinkage of the mortar, will impose a slightly different geometry, and the complex structure will exhibit a complex pattern of cracks. Such cracking may be thought of as natural, and is by no means a sign of incipient collapse. It indicates merely that there has been, at some time, a shift in the external environment to which the structure has responded; indeed, the cracked 'three-pin arch is a well-known and perfectly satisfactory structural form. Moreover, by consulting figure 21 the thrust within the arch, transmitted from voussoir to voussoir, must be located in the extrados at mid-span and in the intrados at the abutments; the cracking engendered by the slight spread of the arch has defined a unique position for the inverted catenary. The fact that the catenary lies within the boundaries of the masonry confirms that the geometry is satisfactory; if the arch did not fall upon decentring, then it will never fall as a result of (small) movements of the abutments (Heyman 2000).

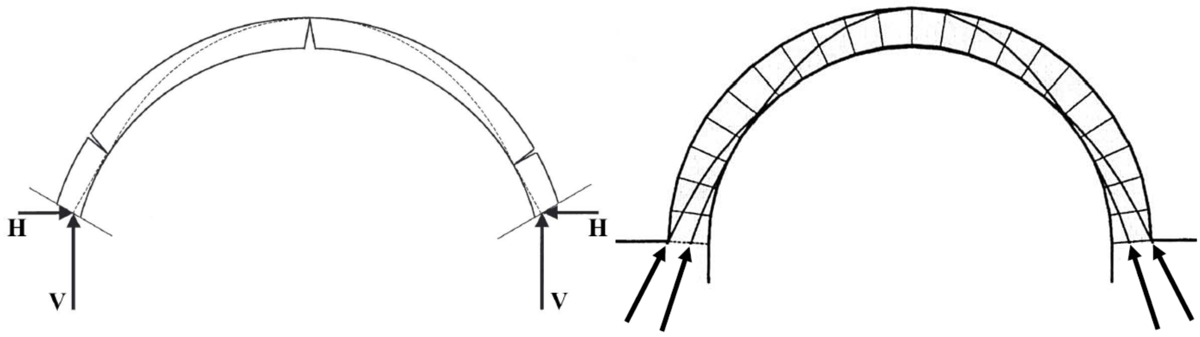



Figure 21 - Internal thrust lines due to self weight of arch (Heyman 2006)

## 3.4 The gothic pointed arch:

Another issue related to the gothic vault, which was clarified in the previous chapter, is the use of the pointed arch and the innovation that it brought specifically to the construction of vaults and generally to the gothic architecture. Heyman, as was explained earlier, studied and explained the behavior of the circular arch. Thus, a question was raised; how different would be the behavior of the pointed arch regarding the stability and its capability of supporting the applied loads.

In 2007, a study for the safety of the masonry pointed arches was conducted by (DeRosa & Galizia 2007). In the study, they evaluated the collapse multiplier of the live loads, or calculated the lowest admissible thickness for assigned loads to the pointed arch. See figure 22.

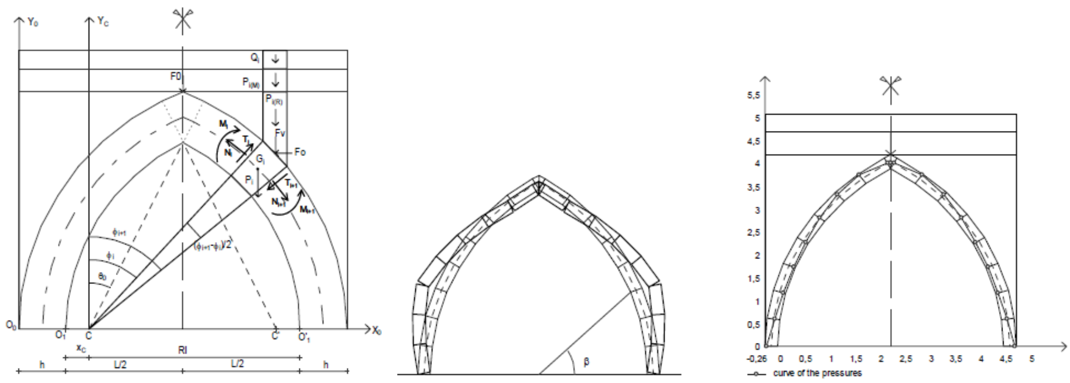



Figure 22 Forces acting on the generic voussoir, collapse mechanism and line of pressure (DeRosa & Galizia 2007).

What can be interesting in the results; was that the highest load's multipliers and the lowest arch depths h were attained for the equilateral pointed arch, meaning that the stability and the safety of an arch grows greatly when the arches verge on a pointed shape and the arch that has a better behaviour is the equilateral pointed arch. Also, the behavior didn't change regarding the span or depth of the arch; however the load multiplier was the one changing. Moreover, by applying a horizontal force to the lowest and again to the third of the height, the load multiplier increasingly rose to reach 100% (DeRosa & Galizia 2007).

In 2010, another study was declared by (Romano & Ochsendorf 2010) regarding the analysis of the structural behavior of Gothic, or pointed, masonry arches. By applying equilibrium conditions, analytical and graphical analyses have been carried out to assess the minimum thickness, the maximum and the minimum thrust, and to compare the behavior of both the circular and the pointed arch. It is to be emphasized that study considered the arch geometry above the level of fill and assumed a uniform arch thickness (Romano & Ochsendorf 2010).

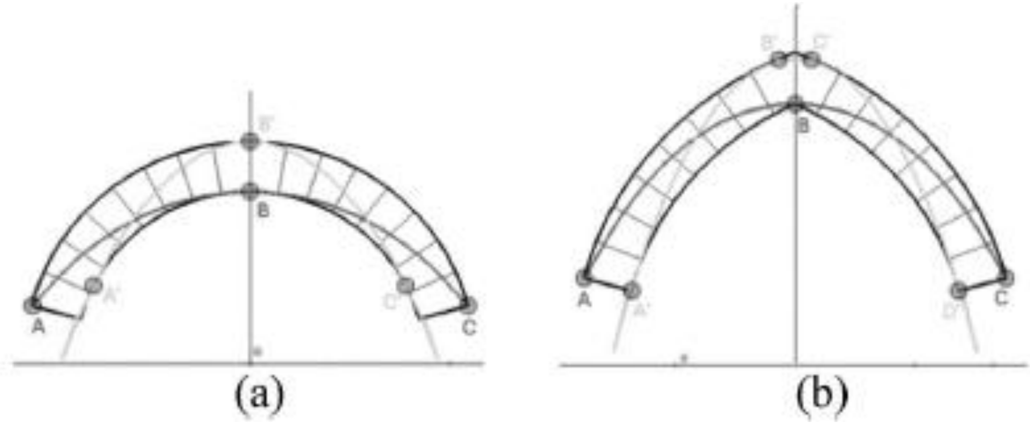



Figure 23 Schematic illustrations showing minimum and maximum thrust lines in a) a circular arch, and b) a pointed arch. (Romano, Alessandra and Ochsendorf, John A. 2010)

In this study, the results were more interesting. First, for the minimum possible thickness, the minimum and the maximum thrust values coincide, giving a unique solution. A thinner arch cannot be constructed without the line passing outside the masonry. In the collapse analysis of symmetric arches under symmetric load, five hinges (semicircular arch or pointed arch with big eccentricity) or six hinges (pointed arch with small and medium eccentricity) must form to give a symmetrical mechanism with one degree of freedom. This is shown in figure 23.

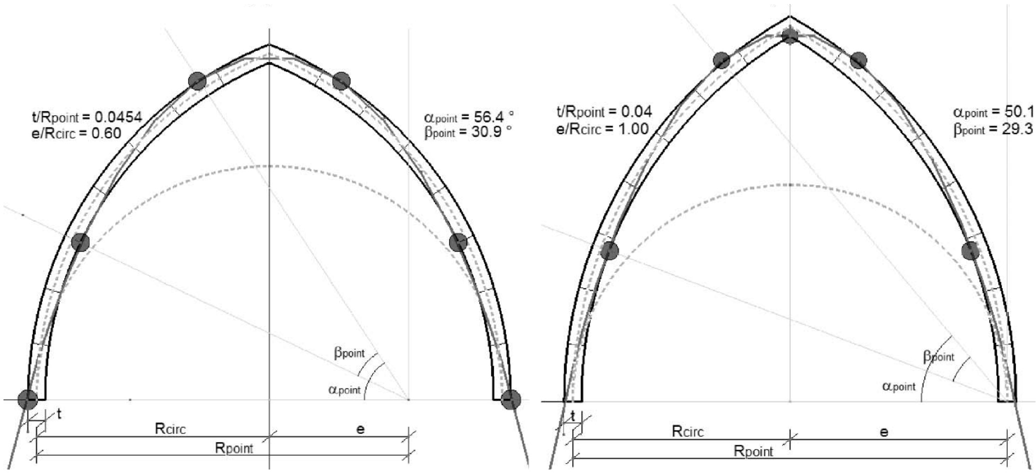



Figure 24 - Schematic illustrations showing minimum thickness thrust line in the pointed arch (Romano & Ochsendorf 2010).

The study succeeded to demonstrate that the pointed arches generally have lower values of thrust than circular arches for most practical geometries. The effect of the rise of the height of the crown leads to a decrease in the maximum and minimum thrusts requiring a greater accuracy in the equilibrium of the supporting structure.

For the collapse mechanism and the ability of the arch to deform before collapse, it was proved that pointed arches on horizontal spreading supports could collapse in two ways; a five-hinge mechanism where the central portion of the arch is a three-hinged arch (symmetrical in the circular arch and asymmetrical for the pointed one with the activation of only one of the two hinges close to the crown) or a three-hinge mechanism by snap-through if the thickness is sufficiently large. Pointed arches are able to deform substantially more than circular arches before collapse. Finally, the pointed arch is capable of bearing greater superimposed loads than the circular arch, especially if the weight is placed at the crown (Romano & Ochsendorf 2010).

## 3.5 Theories for the vault behavior:

### *3.5.1 Robert Willis:*

In 1845, Robert Willis, was the first to study and to write about the construction of the medieval vaults. In his publication "On the Construction of the Vaults of the Middle Ages", he observed that the Gothic ribbed vault "*consists, as is well-known, of a framework of ribs or stone arches, upon which the real vaults or actual coverings of the apartment rest ... The ribs are the principal features, and the surface of the vaults sub-ordinate*." (Alexander et al. 1977) His theory, that the ribs of the gothic vaults play the predominant structural role, was widely spread and accepted.

It is to be noted that Willis himself commented on his work that he mainly centered on the geometry and the construction techniques “form and management” of the vault and didn’t carry out any deep study for its “mechanical principles”. However, he derived his observations within the context of his distinction between what he called mechanical construction (how the weights are really supported) and decorative construction (how they seem to be supported) (Huerta 2009)

Later this theory was criticized by experts as being more based on “faith in the validity of visual impressions than on comprehensive understanding of the vault's structural action”. (Alexander et al 1977)

### *3.5.2 Violet le duc:*

Eugene Viollet-le-Duc, 1858-68, in his *Dictionnaire* (articles *Construction* and Voute) discussed and explained the Roman construction techniques. The Romans developed the use of mass concrete placed in batches in skeletal sections of the vault, abandoning the ideas of 'voussoir' construction. However, when the barrel was complete, and the relatively lighter timber supports were removed, a true 'masonry' resulted. The material was always very strong compared to the small imposed compressive stresses; but weak in tension and any movement apart of the supports, such as walls, would lead to cracking. (Alexander et al 1977)

Later, he discussed the rib vault and explained that the origin of the rib was necessary for the centering for the transverse arches and the groins of the vaults. The wooden centering were replaced by stone-centering that supported the masonry of the webs during the construction and later the whole weight of the vault after the completion of the vault. The ribs were facilitating also the difficult bonding of the stones at the groin. Finally, Viollet-le-Duc foresaw that the loads are entirely transmitted by the ribs to be concentrated at certain points where the inclined flying buttress does transmit the inclined thrust of the vault to the external buttresses. (Huerta 2009)

Viollet also used a new term by that time, the elasticité, which meant the capacity to adapt to the settlements and movements of the supports of the vault. He used this term in his structural interpretation, by understanding the active nature of the internal forces in a gothic structure and the crucial role presented by the equilibrium, that the structure of the gothic building, as the ribbed vault, has the same property of elastcité. This means the global capacity of the structure to adapt to different situations of loads and changes in the boundary conditions. (Huerta 2009)

Later, he derived his theory for the vault that the vaulting acts as independent series of arches supported by the diagonal ribs. (Alexander et al 1977)

### *3.5.3 Victor Sabouret:*

Sabouert, who was a French engineer with a solid formation in applied mechanics and an extensive experience in bridge design, started in 1924 an attack against some of Viollet-le-Duc ideas regarding the ribbed vaults. (Huerta 2009) In Sabouret paper “a provocation to the orthodoxy of gothic”, he argued that the role of the groined ribs mere merely decorative. By explaining his geometrical and materials studies, such as the masonry should work in compression and the sliding is impossible, he clarified that some hinges would occur and two modes of equilibrium would exist. The first, for rectangular bays, he supposed that the thrust may be diffused radially and makes for this an analogy with the skew vaults, and defines for its analysis a series of slicing planes parallel to the front arch and radiating slicing planes for a trapezoidal form.

Enclosing his ideas, Sabouret commented that the dimension of the ribs is usually too small to be of any importance. By remarking the cracking in the vaults, he affirmed his theory that in many cases the ribs are separated from the groin, due to the movements of the vault. (Huerta 2009)

### *3.5.4 Pol Abraham:*

In 1934, Pol Abraham did not accept many of the comments of Violet-le-Duc regarding the structural approach of the vault and revealed many errors in Violet reasoning. Abraham, following Sabouret ideas, made his observations regarding the understanding of the vault behavior and related it to the concept of horizontal thrust. (Alexander et al 1977) First, he studied the planer arch as it consists of materials inclined at various angles with respect to the horizontal, and analysed the forces using his concept of horizontal thrust. He argued that the major forces will be carried at the same inclined angles of the arch inclined materials. He also concluded that neither the type of stone, given equal density, nor the strength of mortar binding the individual stones in any vault arch will change this relationship between vertical and horizontal thrust.

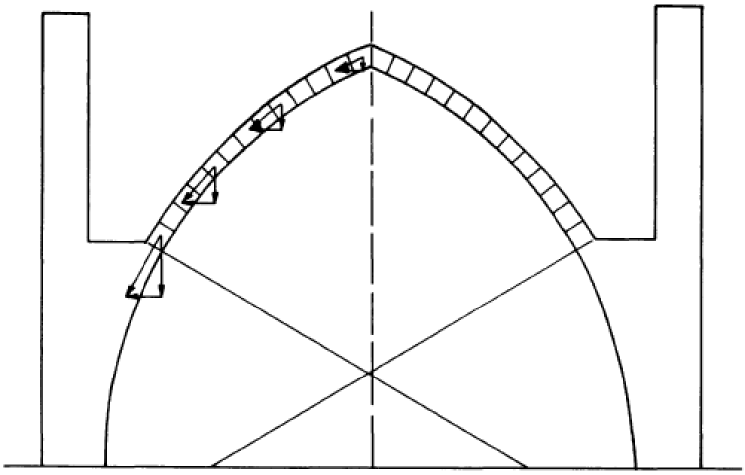


Figure 25 - Abraham’s force polygons for a planar arch (Abraham 1934)

Later, using his analysis for the arch, he explained the behaviour of the three dimensional vault and refuted Violet’s theory. Generally, Abraham mounted a formidable attack to the rational approach to gothic architecture, and he used for this every argument at his disposal. For him the ribs are decorative.  Although his reasoning was not always correct, but the detail of his analysis, the numerous explanatory drawings, and his deep scholarship was convincing for the late researchers to contradict Violet ideas. (Huerta 2009)

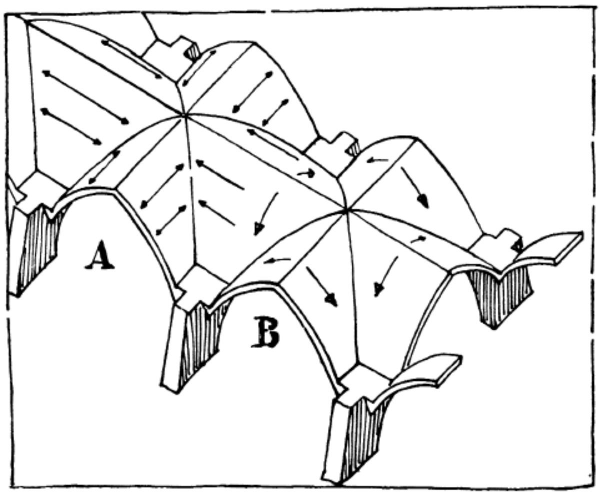



Figure 26- the Vault behavior (from Abraham and Viollet)

### 3.5.5 Jacques Heyman:

Jacques Heyman issued in 1964 an essay about statics of gothic cathedrals, followed it by several papers. In those works, theoretical investigations are refined, and actual applications are developed in order to give definite grounds to the application of the method of limit analysis to the masonry structures. And in 1972 he published his book “the stone skeleton” including a critical edition of the Coulomb essay. Heyman's papers became a main reference for the static analysis of masonry.

Jacques Heyman introduced the first formulation of the plasticity theorems for masonry (Heyman 1966). He successfully applied limit analysis to masonry in the 'standard' format under the assumption that the friction coefficient would be high enough to prevent sliding in the range of problems considered. This powerful tool was specifically applied to the safety analysis of masonry arch bridges and later extended to the analysis of vaulted structures.

For Heyman, the actual stress state cannot be found and is impossible to be known. He abandoned the behaviour's 'quest' to concentrate on the safety and the calculation of the collapse load of the structure. In the spirit of 'standard' limit analysis (lower bound theorem); all methods able to give an admissible stress state can be used to give a lower bound of the collapse load. “*the vault will stand as long as a thrust network can be found that fits within its section*” (Heyman, 1995) with three main assumptions that were accepted in the nineteenth century for a masonry arch analysis and cited by Sabouret (Huerta 2009). They are;

- *Masonry has no tensile strength*
- *Stresses are so low that, effectively, masonry has an unlimited compressive strength*
- *Sliding failure does not occur*

This was explained by the "approach of equilibrium" which concerns the study of possible equilibrium states with the masonry in compression and the existence of these possible states of equilibrium depends on the geometry. The line of thrust, in a masonry arch, represents a possible set of internal forces in equilibrium; if the line is contained within the masonry, the yield condition of the material is satisfied, and the arch is safe. The ability of drawing a line of thrust within the arch means that the arch will not collapse. Any little movement of the abutments will produce a certain cracking and a change in the position of the line of thrust, but due to the Safe Theorem it will never go out of the masonry (Huerta 2009).

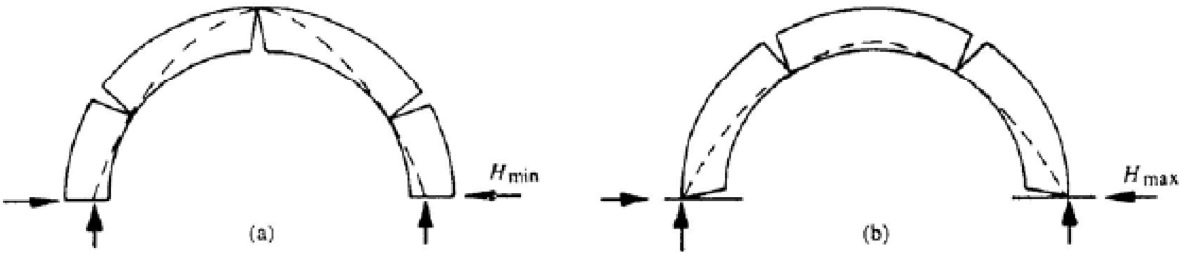



Figure 27 Semicircular arch under its own weight. a) Minimum thrust; b) maximum thrust (Heyman 1995)

Furthermore, Heyman studied the complex vaults and its ribs. His work suggested that the intersection between two shell surfaces develops a large stress concentration (Heyman 1977).

According to (Huerta 2001), Heyman’s modern theory of Limit Analysis of masonry structures is the best tool to understand and analyse masonry buildings. By emphasizing the overall importance of geometry, the theory aims to determining the vaults' safety factor, which can be expressed as the ratio between the geometric thickness over the minimum required thickness, (D’Ayala 2008)

## 3.6 Collapse and cracking of the vault:

The cracks do occur due to the small movements of supports. The developed cracks are not usually a cause of concern and they tell where the forces are not acting. According to (Huerta 2001), the interpretation of the Sabouret cracks developing in cross vaults is that these cracks are not linked to failure but to compatibility problems in the lateral vault and wall connection. This explanation is seen in the light of limit analysis as if due to the development of a one-dimensional collapsing mechanism which is not enough for the collapse. Figure 28 shows the type of non dangerous cracks discussed by Sabouret.

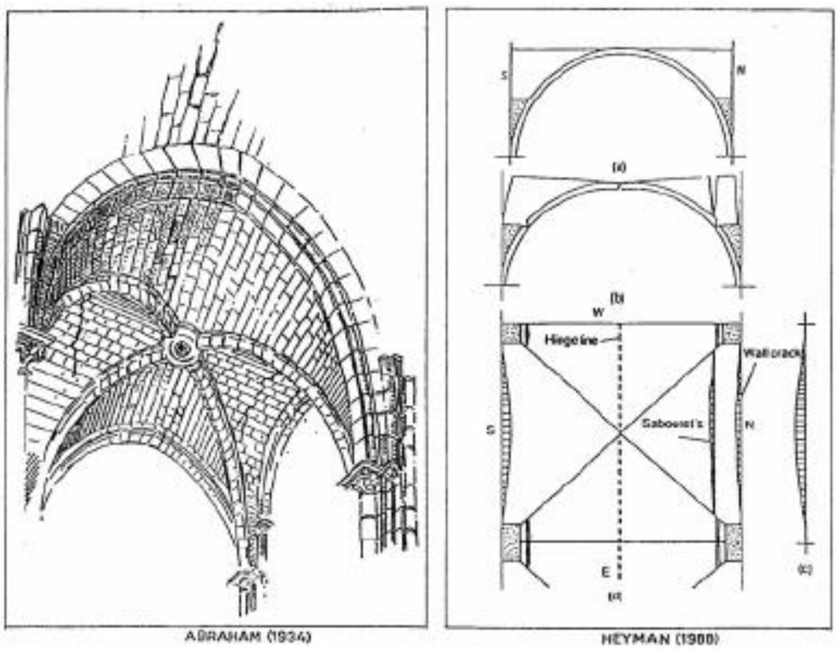



Figure 28 Sabouret cracks in the cross intersected vault (Heurta 2001)

However, cracks can be the cause of collapse when combined with the following actions;

- Displacements; due to foundation movements and mortar creep over time.
- Overloading of the vault; due to water penetration effect and possible collapsing of existing upper roof.
- Accelerations; such as vibrations and earthquakes.
- Vaults’ deformation over time can cause collapse in buildings.

## 3.7 Conclusions:

- Dealing with the historical monuments, and specially the gothic vaults, Stability is the dominant concern rather than failure of the material. Historical masonry structures have very low stress levels.
- Due to the low levels of compressive stress, the failure of historic masonry structures is more likely due to instability rather than to material failure. The strength of a masonry arch depends primarily on its geometry.
- Heyman's modern theory of Limit Analysis of masonry structures is the best tool to understand and analyse historical masonry buildings and thus the gothic vaults.
- The collapse of a masonry arch occurs when the load path can no longer be contained within the masonry. And the determination of the collapse state can be defined based on thrust line analysis using graphic statics.
- Pointed arches in comparison to circular arches do exert lower horizontal thrust values and resist greater support displacements. Medieval masons clearly appreciated these structural advantages.
- A strong debate have been going on for decades about the structural behavior of the gothic vaults, especially the ribs function and the system of transmission of loads in the vault. Although, it was never analytically confirmed, many researchers are convinced that the ribs has no structural role, however they admit the existence of an increase of the stresses at the rib.
- The development of cracks in the vault doesn't mean an occurrence of collapse, but cracks accompanied by one or several intimidating actions can definitely lead to collapse.

# 4. ANALYSIS OF THE VAULT

The first ‘‘scientific” graphical attempts for the study of the equilibrium of curved masonry, such as vaults and domes, go back to the early 18th century and are due to, e.g. Bouguer (1734), Bossut (1778) and Mascheroni (1785), who stated simple mono dimensional equilibrium equations, neglecting the role of circumferential forces. It was clear from the beginning was that cracking occurs on curved masonry elements in presence of self-weight and of very low tensile stresses. In this context, a considerable improvement in the analysis of spherical domes was achieved when Levy (1888) proposed a graphical analysis aimed at finding the circle on which circumferential forces vanish. For an exhaustive history of the theories of masonry vaults we remand to the classical treatise of Benvenuto (1991). Nowadays it can be affirmed (Huerta, 2001) that ‘‘the modern theory of limit analysis of masonry structures, which has been developed mainly by Heyman (1969), is the tool to understand and analyze masonry structures”. (Milani 2008)

While craft traditions had sufficed for the remarkable traceries of the Gothic construction, theoretical explanations were sought in the Renaissance, see Elliot (1994). Leonardo da Vinci could have been the first to contend that the thrust followed a path that remained within the arch, but much of the lengthy study that commenced in the Renaissance focused on the construction of domes. In experiments, chains were draped to represent the curves that might be the inverted lines of thrusts and intricate graphic solutions attempted to follow forces from stone to stone. Long after Leonardo, in 1586, Simon Stevinus published a book on statics; its translation to Latin in 1608 as “Mathematicorum Hypomnemata de Statica” made his knowledge accessible to scientists and mathematicians throughout Europe, and provided the basis for nineteenth century graphic statics, which enabled the solution of structural problems through drawings. (Lourenço 2001)

Today, several methods and computational tools are available for the assessment of the mechanical behaviour of historical constructions. The methods resort to different theories or approaches, resulting in: different levels of complexity (from simple graphical methods and hand calculations to complex mathematical formulations and large systems of non-linear equations), different availability for the practitioner (from readily available in any consulting engineer office to scarcely available in a few research oriented institutions and large consulting offices), different time requirements (from a few seconds of computer time to a few days of processing) and, of course, different costs. It should also be expected that results of different approaches might be also different, but this is not a sufficient reason to prefer one method over the other. (Lourenço 2001)

Computational approaches to the integrity assessment of structural masonry are presently conducted by a variety of methodologies, ranging from highly simplified methods to complex non-linear finite clement analyses using plasticity based material models, including joint and interface elements to model planes of weaknesses.

## 4.1 Types of analysis:

Historic masonry structural analyses are carried out using idealisations, which should definitely be appropriate to the problem under consideration, for both the geometry and the behaviour of the structures. (Lourenco 2001) First, idealization of the geometry is needed due to the complexity of the geometry of historical masonry structures and the difficulty of distinction between decorative and structural elements. Thus, the geometric idealisation should be kept as simple as possible. Second, due to the fact that masonry is a complex composite material made of units and mortar and having a very low tensile strength, idealisation of the masonry behaviour is required. Typical masonry behaviour idealisations are the elastic behaviour, the plastic behaviour and non-linear behaviour.

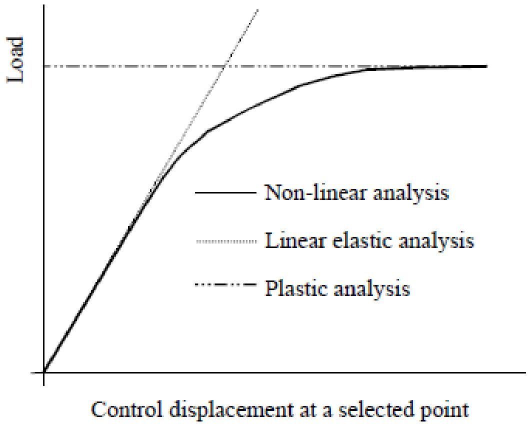



Figure 29 General load-displacement diagram for a structural analysis. (Lourenco 2001)

Among the types of analysis for the historic masonry structures, the limit analysis comes on including the graphical analysis, kinematic and limit state analysis. Also, the finite element analysis

### *4.1.1 Limit analysis:*

The limit analysis, which is also called plastic analysis, aims to evaluate the structural load at failure. The material of the structure must exhibit a ductile response and can be assumed as adequate for the analysis of historic masonry structures if a zero tensile stress is assumed.(Lourenco 2001)

As demonstrated by Heyman (1966), Masonry structures can be analysed as an assembly of rigid blocks using limit analysis, in which the minimum and maximum thrust states can be determined from static equilibrium. Heyman also suggested two governing limit analysis theorems for masonry, the lower and upper bond.

The Lower Bound Theorem confirms that if it is possible to demonstrate at least one possible equilibrium state, then the structure can also find at least one possible stable state. Following the theorem, the analysis seeks permissible line of compressive force for the given loading.

The Upper Bound Theorem proves that when the load path can no longer be contained within the structure, and it is the unique and largest possible load, then it is the collapse load for the structure, and thus seeks finding the critical load which results in a failure mechanism.

Limit analysis can be used to determine collapse states based on thrust line analysis. This approach represents an effective tool for understanding the main aspects of the ultimate behaviour and evaluating the ultimate capacity of the historical constructions.

#### *4.1.1.1 Graphical (line of thrust) analysis:*

It is a static plastic analysis method that is based on the lower bound theorem and the static equilibrium. Generally, it aims to approximate the forces which hold the structure in equilibrium. It is also used for the calculation of the geometric safety factor which represents the ratio between the actual thickness of the arch and the minimum thickness of an internal arch with the original span and able to resist the original applied load.

As for the analysis of complex vaults, as in the case of Gothic vaults, it can be conducted following the safe theorem of Heyman, meaning the application of the equilibrium approach. According to (Huerta 2001) the approach was first suggested by Frézier (1737) and it was applied many times since (for example by (Dietlein 1823), in the context of (La Hire's incorrect, theory of arches). Later, Heyman (1966, 1977) showed the systematic within the frame of Limit Analysis.

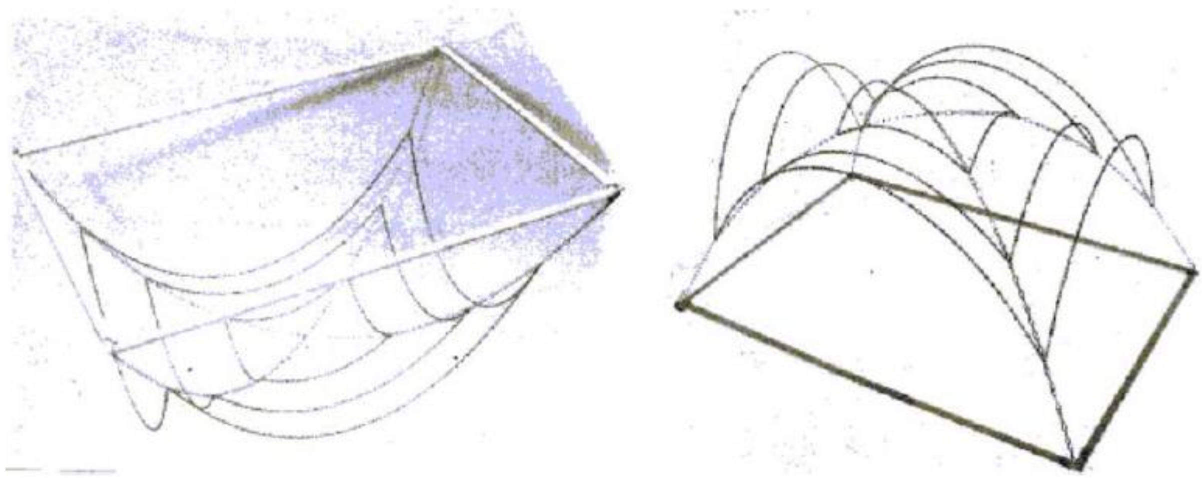


Figure 30 Limit analysis of a gothic vault (Heyman 1995)

The technique explained by Huerta, consists of dealing with the vault as it is formed of a series of elemental sliced arches which rest upon the cross ribs. The pattern of slicing will depend on the form of the vault. Then, is the searching and calculation of the possible line of thrust in every one of these sub arches and finding a possible equilibrium solution in compression. Later, it is the analysis of the cross ribs under the system of loads formed by the reactions of every elemental arch. Accordingly, the structure can be determined as safe by the Safe Theorem. The process can be called the slicing technique.

However, it was mentioned by (Lourenco 2001) that the application of the graphical analysis method to larger structures is rather cumbersome and the issue of the determination of the structural safety is difficult to solve.

### *4.1.1.2 Kinematic analysis:*

The kinematic analysis is based on the upper bound theorem as yield hinges for arches. The method is applied to systems composed of rigid blocks and consists of different steps. (SAHC 2009) First the rotation centers for the studied arch are identified as the movements of individual blocks occur as shown in figure 31. Then the work done by external forces is calculated and applied following the upper bound theorem. Thus, the method is useful for the determination of the upper bound of the vertical live load, and also to determine a seismic coefficient for the arch. Kinematic analysis can provide the assessment of structures composed by arches as shown in figure 32.

Kinematic approach only provides an upper bound of the real ultimate load, and in order to determine the real ultimate load, either a mathematic optimization method (leading to a minimal value) or a process based on successive rough estimates are needed. (SAHC 2009)

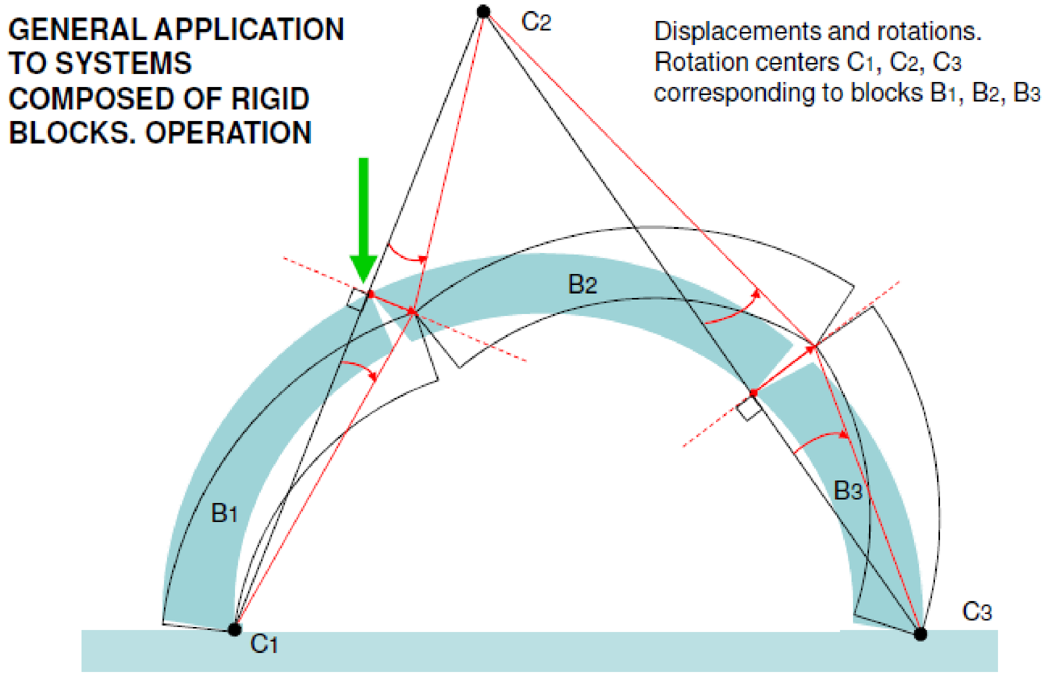



Figure 31 kinematic analysis of an arch with rotational blocks (SAHC 2009).

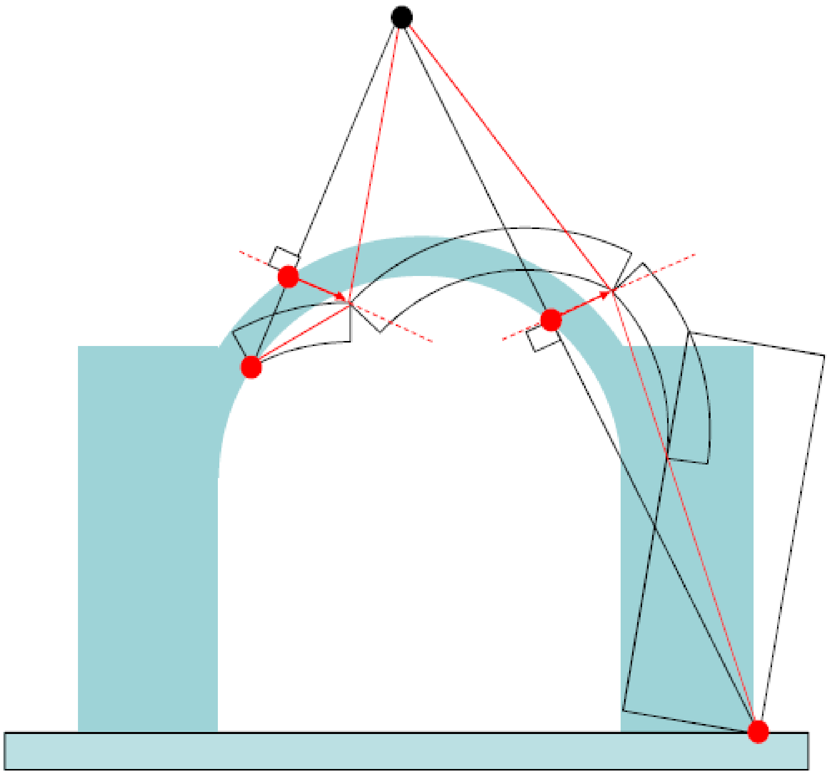


Figure 32 assessment of arch structure, the weakest case and frequently developed failure mechanism (SAHC 2009).

#### *4.1.1.3 Limit state analysis with finite friction:*

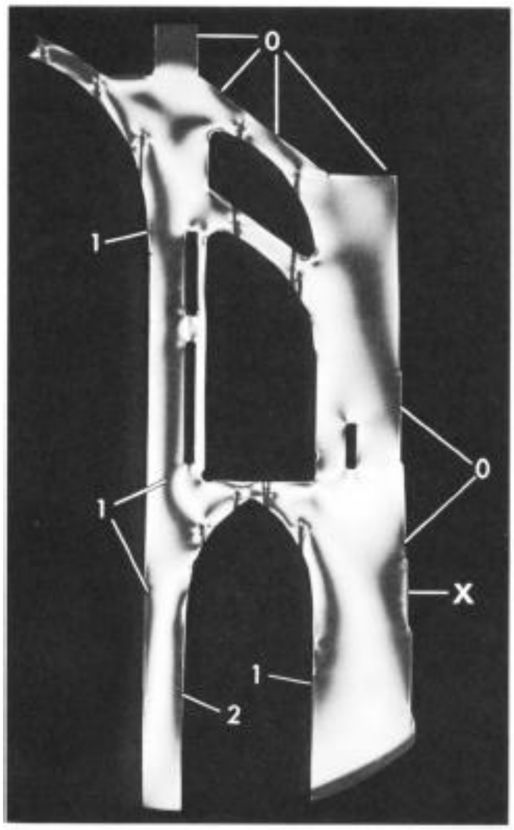
D'Ayala proposed a different concept of limit state analysis with finite friction. This is a computational procedure which allows defining the 3D structural behaviour of masonry (D'Ayala 2008). The proposed analytical method, based on lower bound approach, allows obtaining, for a generic type of vault, the actual crack pattern, the stress field and the horizontal thrust at the supports for both gravitational and localized loads. Limit analysis has the substantial advantage of disregarding intermediate states of stress in order to identify directly ultimate conditions. It tackles the problem of main interest to structural engineers: control of safety levels. The assumption is that at failure pre-existing self equilibrated states of stress are irrelevant so that direct correlation can be established between external actions and collapse mechanisms. However these assumptions arc only valid for 'standard' materials for which the general theorems of plasticity apply. Due to its lack of ductility and presence of friction, masonry is not 'standard'.

Through this approach (D'Ayala 2008) tries to understand the structural behaviour of pavilion vaults, in order to evaluate with more accuracy the actual stress field that can cause cracks. Moreover, he claims that his method allows finding the thrust line position and the admissible thrust surfaces, leading to the calculation of the minimum thickness which satisfies at the same time equilibrium and compatibility.

The limit state analysis with finite friction constitutes a new tool to define the 3D structural behavior of masonry vaults. By means of an easy and rigorous procedure, developed with commercial programs (for example Excel), it is possible to obtain an optimum solution considering the load redistribution after cracking. It allows determining the vaults' safety factor, and also the accurate position of the hinges at failure and hence the appropriate positioning of ties or other thrust contrasting devices.

### *4.1.2 Photo-elastic analysis:*

The photo-elastic method, proposed in 1968 by Robert Mark, consists of building very small models assembled from precisely machined epoxy plastic components that might be loaded in an oven where, at specified temperatures, the model materials would change to a highly sensitive rubbery state. Model deformations in this state are irreversible and thus are maintained or "locked in" when the oven temperature is slowly lowered. Following cooling, and removal of the loads, the fixed deformations can easily be measured to give an overall picture of the model's behavior. (Mark 1968)



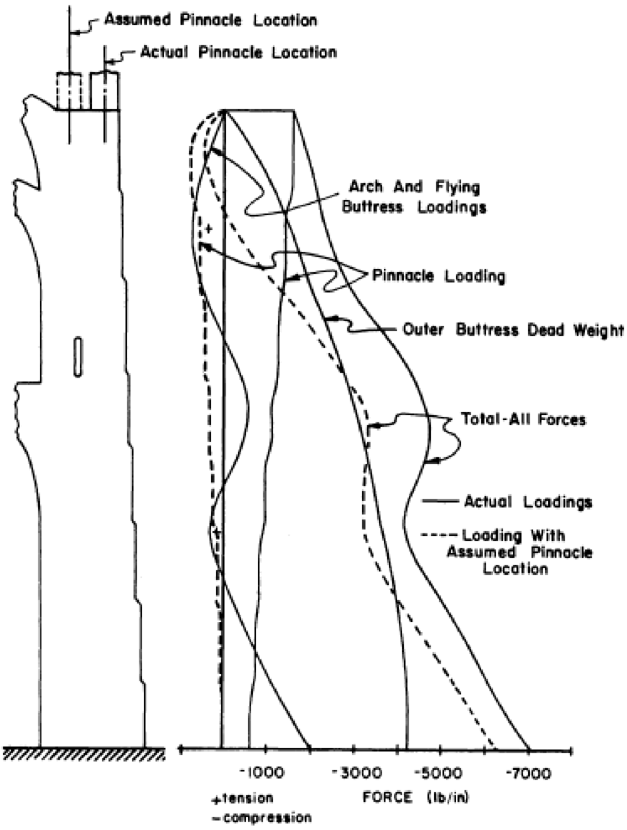




Figure 33 Mark photoelastic fringe pattern

Figure 34 force distribution for the analysed section

Figure 33 shows the fringe pattern observed in a crossedfield circular polariscope with monochromatic light. Fringes are related to internal force distributions in the manner of a contour map. The predicted distribution of forces in the cathedral along the outer surface of the buttress and under the modeled loadings is shown in Figure 34. The full model analysis revealed the region of the section experiencing tensile forces, high stress concentrations and identification of the point of maximum compressive stress.

Again, in 1973 another comprehensive three-dimensional structural study of quadripartite vaulting, was carried out by Mark, Abel, and O'Neill, and was based on a combination of small-scale photo-elastic and numerical computer modeling.(Mark et al 1973) Two bays of the I3th-century choir vaults at Cologne Cathedral were constructed at I/50- scale from stress-free, cast epoxy plastic that can be seen in Figure 35. The actual vault form was altered only slightly; each vault segment was taken as part of a surface of a shell of revolution, allowing the model to be assembled from four standard components, right- and left-hand longitudinal and transverse webbing.

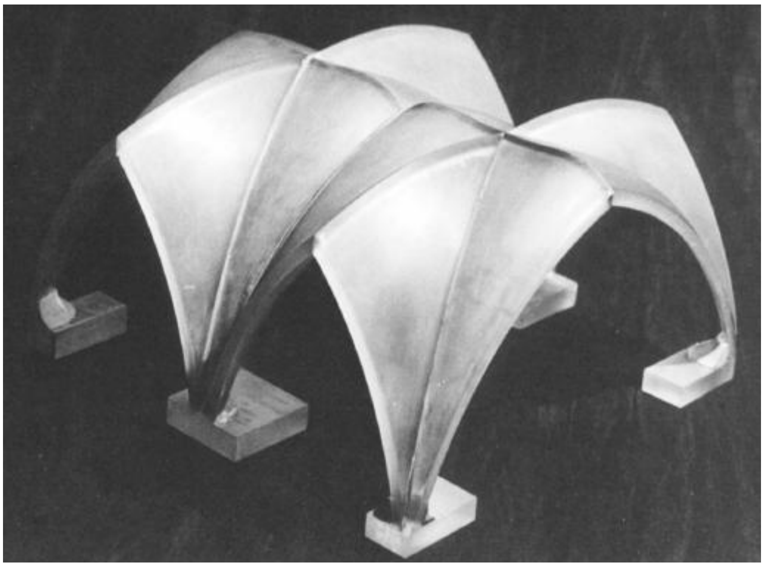


Figure 35 Photoelastic model of Cologne vaulting.(Mark 1977)

The finished model was then loaded in an oven as normally prescribed for a "stress-freezing," photoelastic test. Following this cycle, the model was observed in a polariscope to study "locked-in" photoelastic patterns, and then slices were taken from it to reveal internal stress distributions. The structural action of the model, including magnitudes, distributions, and directions of structural forces as well as support thrusts, was quantitatively determined from these data and used with scaling theory to predict the structural action of the full-scale vault system.

The Mallorca cathedral was one of the case studies Mark worked on in 1982 (Roca 2001) and analysed using the same technique. Roca concluded that this study had some of the conclusions that were not in agreement with the graphical studies of Rubio. However, the photoelastic study succeeded to predict interesting unique results of the compressive stresses state for this cathedral relative to the other Gothic churches discussed by the author.

Figure 36- Photo-elastic analysis showing distribution of internal forces (Mark 1982 after Roca 2001).

### *4.1.3 Elastic analysis (FEM):*

Elastic analysis by means of finite element modeling is as a modern analysis tool and assumes that the material obeys Hooke's law. This is hardly the case of masonry under tension, which cracks at very low stress levels and cannot be appropriate for the historic structures.

The FEM elastic analysis method was firstly adopted by Robert Mark when he combined the use of spatial models with the employment of Finite Element programs trying to extract conclusions of the behaviour of gothic vaults and structures and he published many articles on the subject. Using both techniques, Mark has studied the gothic ribbed vaults. The results were eventually compiled in a book (Mark 1982). Other researchers as Kübler (1974) followed the same elastic FEM methods. (Huerta 2009)

The method was criticized to the fact that Historical masonry construction cannot be conceived as made of a continuous, isotropic elastic material. It is even harder to consider the possibility of knowing the compatibility equations, both internal as the connection of different and external as the boundary conditions, which control the geometry of deformation (Heurta 2009). The obtained solution by the analysis is not the "actual" solution, that represents the "real state" of internal forces yet it provides a possible solution, between the infinitely many possible in such highly hyperstatic structures.

Heyman has pointed out many times, it is a fact that the system of the equations of equilibrium, material (constitutive) and compatibility, is extremely sensible to small changes in the compatibility conditions, particularly in the boundary conditions (Heyman 2008). This may be readily checked using the same FEM packages: a settlement of a few centimetres in a column, a leaning of 1° of a buttress, will distort completely the "actual" state obtained some minutes before. As it is impossible to know the internal and external compatibility conditions, which, besides, change with time, the classical elastic approach is nonsensical.

Elastic analysis cannot tell about the collapse state of masonry and cannot be used to describe the actual state of a historic masonry structure. Researches done by (Block et al 2006) showed that finite element analysis, calculating elastic stress patterns, couldn't determine the failure of different arches. However, Finite Element computer programs is considered by many historians and engineers as the best tool to investigate the behaviour of historical masonry constructions now by.(Huerta 2009)

### *4.1.4 Non linear analysis (FEM):*

Lourenco confirms that the non linear FEM analysis is the reference analysis that should be primarily considered for understanding the behaviour of historical masonry constructions (Lourenco 2001). Also Huerta supports the idea by explaining the method as a simulation of a material which has no tensile strength and which is much realistic than the elastic approach. (Huerta 2009) But the fact remains that the obtained solution may not represent the "actual" state of the structure. The system of equations remains to be very sensible to the original boundary conditions and is also sensible to the history of loading. However, interesting results may be extracted from the use of non-liner FEM packages. Barthel (1991) made a comprehensive study of the behaviour the possible crack patterns in cross vaults, combining the use of the computer.

## 4.2 Evolution of the Graphical static analysis:

During the 1870's the graphical analysis methods evolved, mainly for masonry arches, in combination of the strip method, and made possible the load bearing system analysis of the masonry vault or dome. Later, the graphical method was used for the purpose of design for many constructions. Graphical analysis achieved a remarkable success in the investigation of historic structures especially the gothic.

### *4.2.1 Wittmann (1879):*

In 1879, Wittmann published his graphical analysis which was probably the first correct graphical analysis of three dimensional masonry constructions such as cross (rib) vaults. (Huerta 2008) and Kurrer 2008)[1]

[1] Kurrer,K-E, The history of the theory of structures, form arch analysis to computational mechanics, 2008.

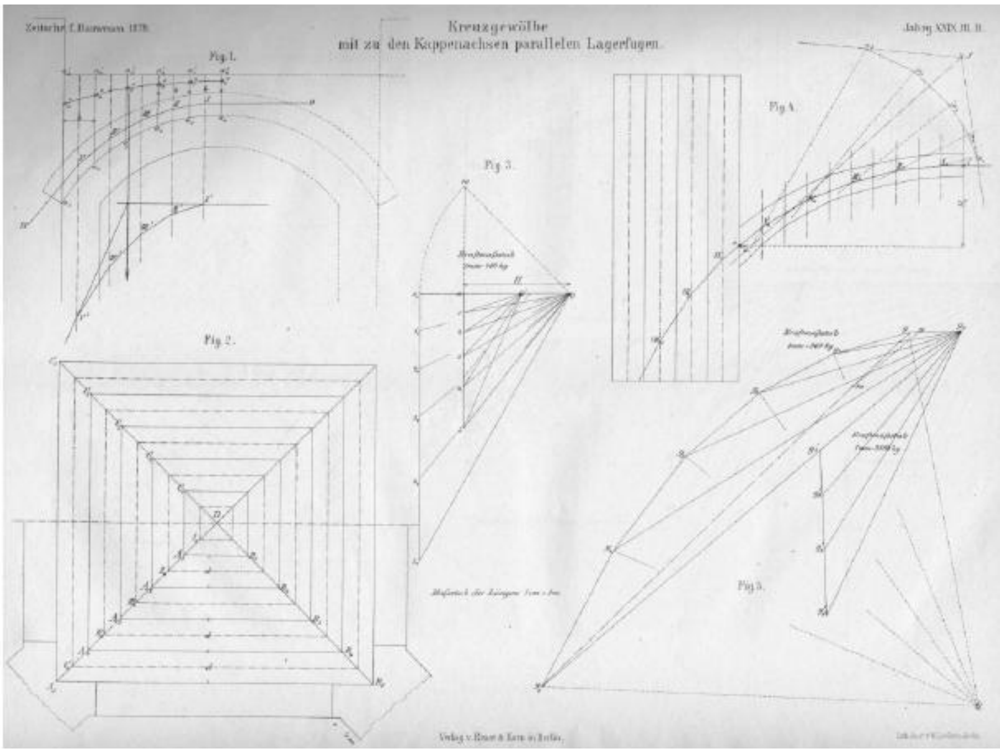


Figure 37 the 1st known application of thrust-line analysis combined with the slicing technique by Wittmann

### *4.2.2 Mohrmann (1890):*

Karl Mohrmann wrote “the textbook of gothic constructions” in 1890, in which he was dealing particularly with the rib vaults of neo-gothic churches. His writing is counted as the most complete presentation of structural investigations of stone vaults according to (Huerta 2008). This work was explained in the first two chapters, which were on masonry vaults and buttresses, and which contain the best presentation of the construction and structural analysis of gothic masonry structures. Mohrmann first describes the application of strip method for determining the resulting compressive forces in a gothic vault: “*the vault segments are divided into strips, the form of which depends on the shape of the vault. The resulting compressive forces due to the vault strips then act on the ribs of the vault, are transferred from these to the buttresses as support reactions, and are carried by the buttresses down to the foundations together with other actions, e.g. the self weight of the buttress*.” This is shown in figure 38.

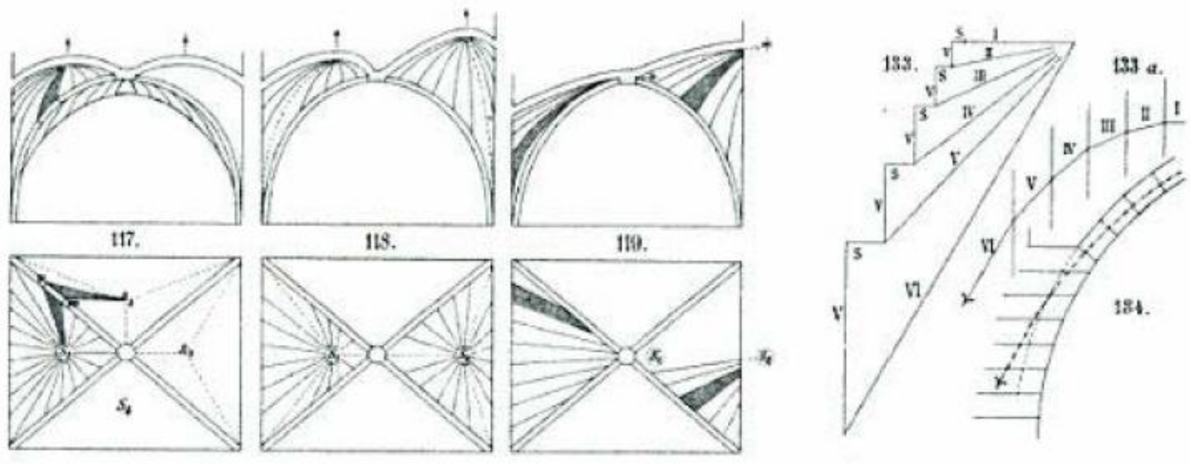


Figure 38 Mohrmann's possible division of the vault segment into strips- the line of thrust in the rib vault. (Ungewitter 1890)

As the previous method involved tedious calculations and graphical analysis operation, especially when the division of the vaults implies many different shaped slices, it was mentioned that Mohrmann proposed another more rational method; "he calculated the horizontal thrust(per square meter of plan area of the vault segment of the rib vault generating a load) for different materials, loads and rise ratios from the global equilibrium for the half vault of the individual vault span, and compiled these in a table together with the intersecting parameters." It is to be noted that this table is still useful for the structural re-analysis of rib vaults. (Kurrer 2008)

Mohrmann applied his slicing technique considering different complicated forms and combinations of vaults (Huerta 2009). He tried to follow some kind of law to decide the combination of cutting planes which will divide the vault's web in elementary arches. He considered that the forces will follow a path similar to that followed by a ball rolling down on the extrados of the webs, which is shown in figure 39, just to imagine the pattern of cutting planes.

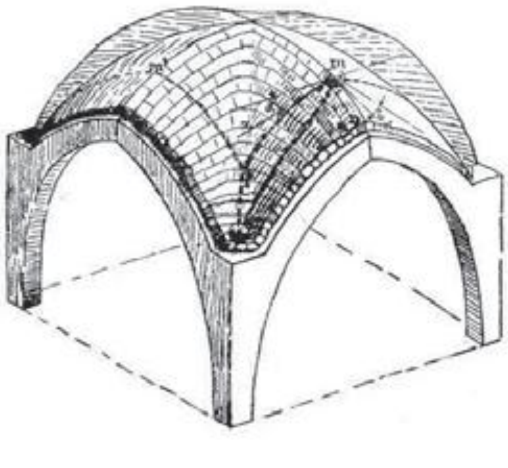


Figure 39 - Path of forces: rolling ball principle. (Abraham 1934, after Huerta 2009)

### *4.2.3 Guastavino (1892):*

Rafael Guastavino Jr. designed using equilibrium methods of analysis and his contributions to the graphical analysis of domes are particularly interesting. He was among the first to adapt new innovations in the use of graphical methods for his design and construction projects.

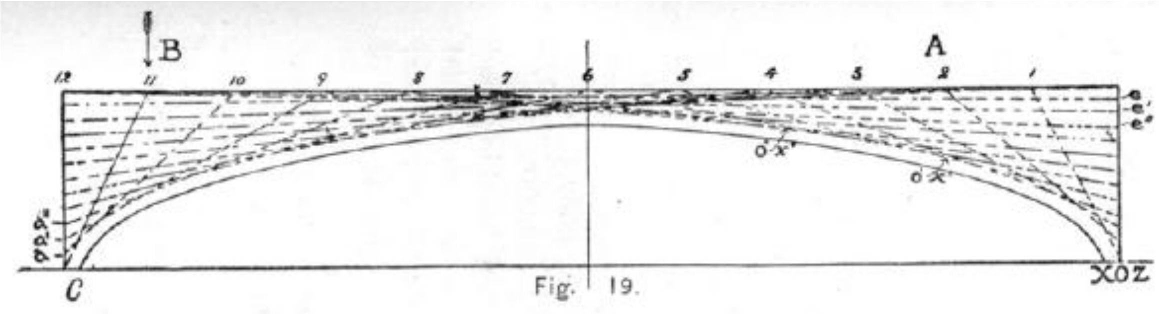



Figure 40 illustration of the live load carried on a tile vault with backfill (Guastavino 1892)

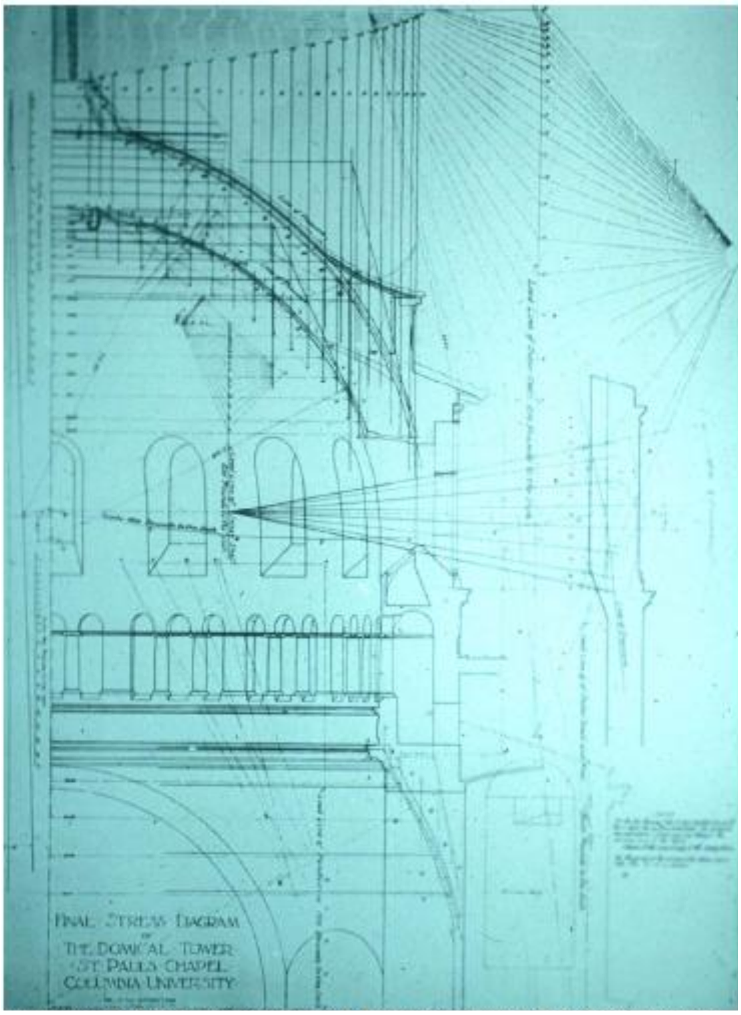



Figure 41 Graphical calculations by Guastavino Jr. for a thin masonry dome (

### *4.2.4 Sondericker (1904)*

Jerome Sondericker wrote in 1904 his book “Graphic statics, with applications to trusses, beams and arches”. He was an experienced teaching professor of graphic statics at the Massachusetts Institute of Technology, and he aimed in his book to deal specifically with the problems encountered in building construction and design. He gave a detailed explanation of the use of graphical analysis and its implementation on the masonry arches.

First, he defined the graphical analysis that it implies the use of graphic statics as having for its object the solution of problems in statics by means of geometrical constructions; the results are obtained directly from the scale drawings.[2] The concept of the analysis consists of the representation of forces by their magnitude, direction and point of application then the resultant of the system of forces within the same plane is determined either geometrically or algebraically and finally the check for the equilibrium of the system of forces. (Sondericker 1904)

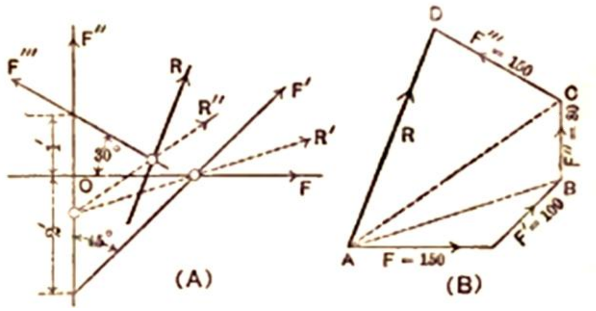
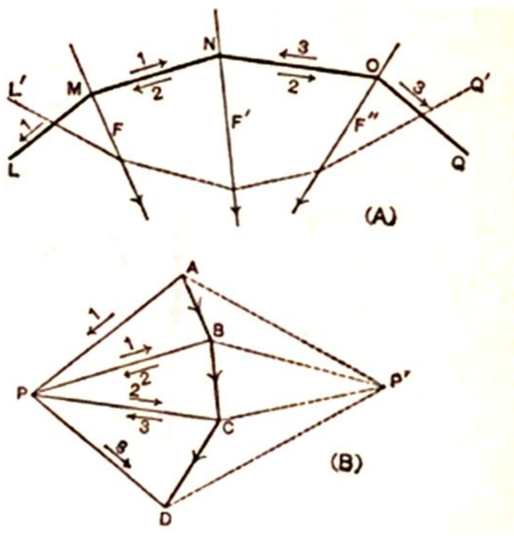



Figure 42 graphical resultant of forces (Sondericker 1904)

The geometrical determination for the resultant of forces can also follow a funicular or an equilibrium polygon as shown in figure 42.




[2] Sondericker,J. , 1904, Graphis statics, with applications to trusses, beams and arches, John Wiley and sons, New York.

Figure 43 Funicular or equilibrium polygon (Sondericker 1904)

The algebraic determination implies resolving each force into two components following two perpendicular axes then calculating the sum of the axis components as $R_x = \Sigma F_x$ and $R_y = \Sigma F_y$ and the resultant of forces will be $R = \sqrt{R_x^2 + R_y^2}$ with a direction angle $\tan \alpha_r = \frac{R_y}{R_x}$. The condition of equilibrium is checked by the sum of each set of components should be equal zero and the sum of moments should also equal zero.

The use of the graphical analysis can be used for the analysis of the masonry arch. The arch is divided in multiple blocks. The load supported by the block and its own weight are graphically represented with a line of action passing by the center of gravity of the block. A balance of forces will exist due to two forces exerted by the adjacent stone blocks. As shown in figure 44, a triangle must be formed by these forces and their lines of action must intersect at the same point.

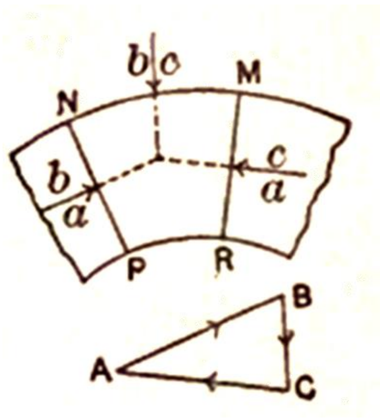



Figure 44 graphical analysis of a stone block. (Sondericker 1904)

Sondericker went forward to explain what he called “line of pressure” (recently called the line of thrust by Heyman). As shown in figure 45, the lines of action of the resultant pressures on the successive joints of an arch are the strings of a funicular polygon, the corresponding rays representing the magnitudes of these pressures.

He named the funicular polygon as the line of pressure of the arch and defined it as “*the broken line joining the centres of pressure of the successive joints*”.

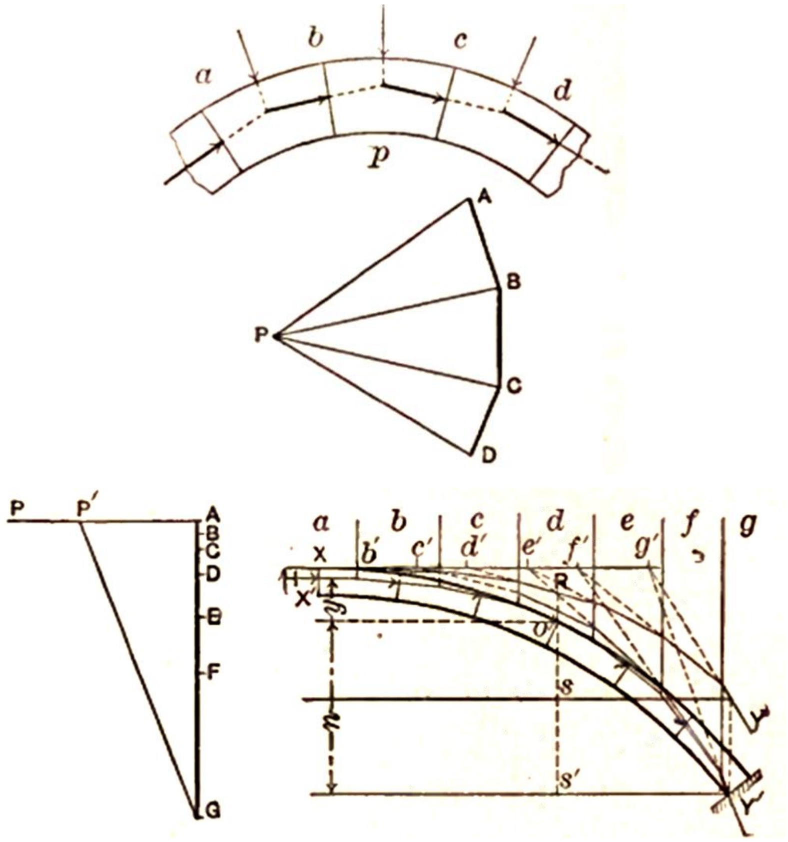



Figure 45 determination of the possibility of drawing a funicular polygon (Sondericker 1904)

The aim of the use of the line of pressure (line of thrust) is to define whatever the arch is stable or not, meaning that the arch can support the subjected loads and its own weight. He expressed his idea as follows;

"*If, then, it is found impossible to draw any funicular polygon for the given system of loads which will satisfy this condition, the proposed arch is unsatisfactory*."

Moreover, he also defined three conditions of stabiltity to be taken into consideration for the analysis of the arch blocks which are the following;

*"(i) The block must not overturn about an edge, as N; (2) it must not slide over the joint; (3) the material of the stone and mortar must not crush."*

Sondericker also used the term of true line of pressure and related it to the form of arching. He favored, to insure stability; that the true line of pressure coincide, as nearly as possible, with the centre line of the arch-ring. And he mentioned a theory of least crown pressure for its allocation. This theory is based on an observation that most arches settle at the crown when the centering is removed as a result of rotation and not the sliding of the arch stones.


Figure 46 Deformations of the arch due to rotational settlement (Sondericker 1904)

### 4.2.5 Antoni Gaudi (1852-1926):

Gaudi used hanging models and graphical methods as design tools according to (Huerta 2006). These methods can be traced back to the end of the 17th Century. His innovation was not the use of equilibrated, catenarian forms, but was the idea of basing all the structural design in considerations of equilibrium. Gaudi also employed unusual geometrical forms for some of his vaults and ruled surfaces, showing a deep structural insight. Also, he designed tree-forms of equilibrium for the supports of the vaults in the Sagrada Familia (Huerta 2006).

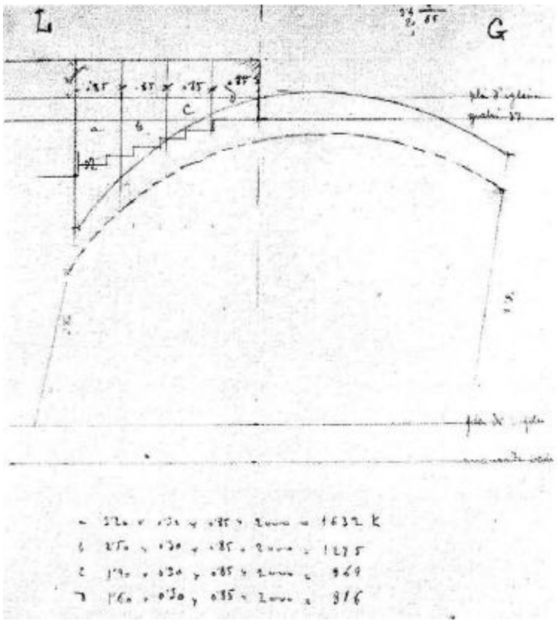


Figure 47 Draft of Gaudi calculations for the design of one of the arches for the church of Colonia Guell (Huerta 2006)

Antoni Gaudi (1852-1926) was a Master Builder. His work covers all aspects of architecture: layout, ornamentation and stability. Any study of Gaudi's work must embrace this global concept of the project. One of the aspects of Gaudi's work was the design and calculation of structures. For Gaudi, structural design was an integral part of architectural design from its initial stages. It was not restricted, as was the usual case in those days, to a mere stability check. This was realized by using graphical methods for the design of "equilibrated" masonry structures, such as the design of the crypt of the church of the Colonia Güell and park Guell in Barcelona. Another example of the use of funicular cables for his design of 3D building structures is the famous unfinished `masterpiece' in Barcelona, the Sagrada Familia that was designed using cable models.

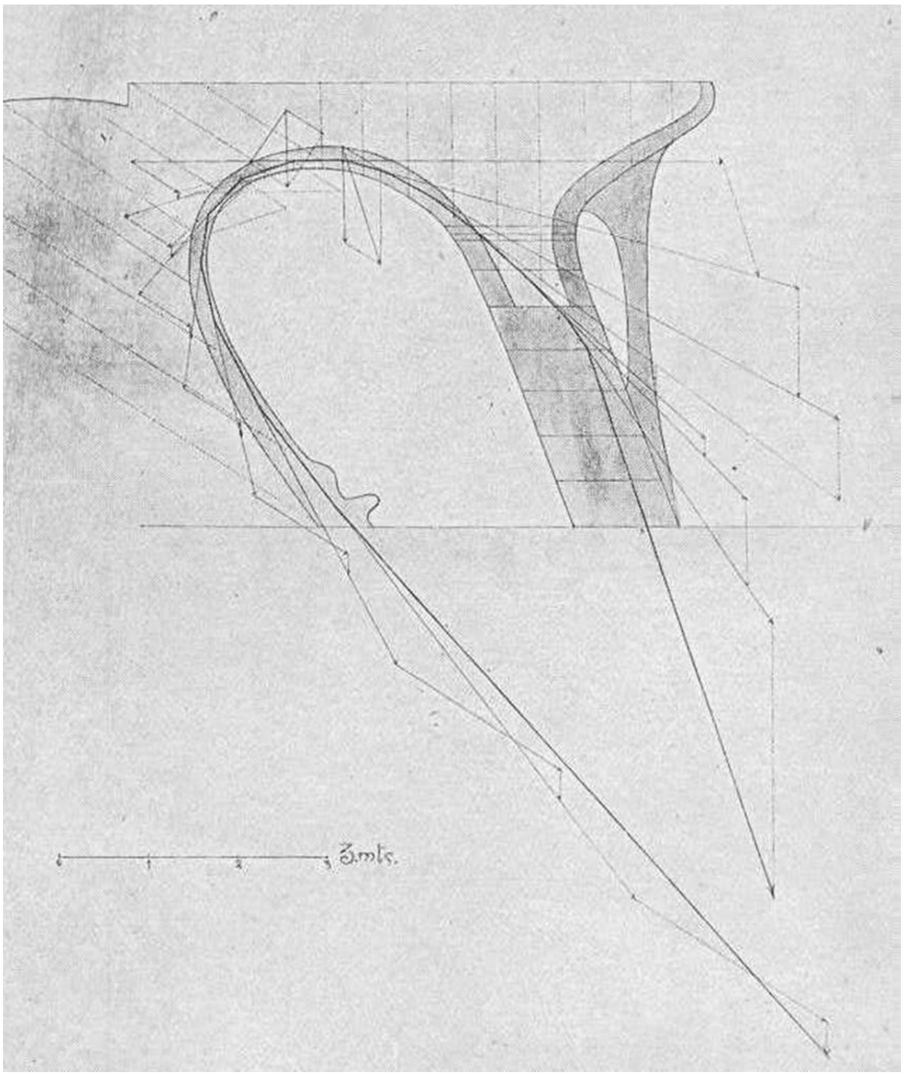



Figure 48 - Gaudí's graphical design for the columns and retaining wall of the Park Güell (Rubió 1913).

Huerta also concludes that the combined use of graphic methods and models allowed Gaudi to obtain a deep understanding of stability and shape problems in masonry arches. (Huerta 2006)

### *4.2.6 Klaus Pieper (1950):*

According to (Huerta 2009), Klaus Pieper used the classical analytical techniques of graphical analysis involving the slicing technique and the drawing of thrust lines, as other engineers for the consolidation and rebuilt of the historical monuments after the severe destruction caused by the second world war. The notable contribution of Pieper lays in his method of using graphical equilibrium analysis, considering the real geometry with its leanings and distortions, and the real state of the masonry, with its cracks for the reconstruction and consolidation of many German churches as in the case of

Marienkirche in Lübeck. (Figure 49) His life long experience was compiled in his book "Sicherung of historischer bauten".

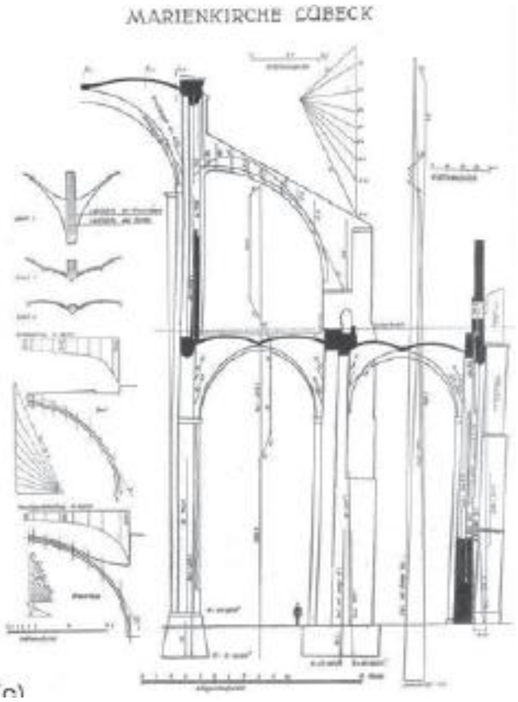



Figure 49- Pieper's Statical analysis of the Marienkirche, Lübeck (Huerta 2009)

### 4.2.7 Philippe Block (2005):

In a thesis submitted by Philippe Block in 2005, he proposed a new analysis tools using thrust lines specifically for vaulted masonry buildings, which are also interactive learning tools for historians and architects (Block 2005). He used thrust lines to illustrate possible collapse modes and to allow users to clearly visualize the forces within the masonry.

Using his technique, he analyzed different elements such as 'Landscape Arch' in Devil's Garden, Arches National Park in Utah, USA (Block 2005).

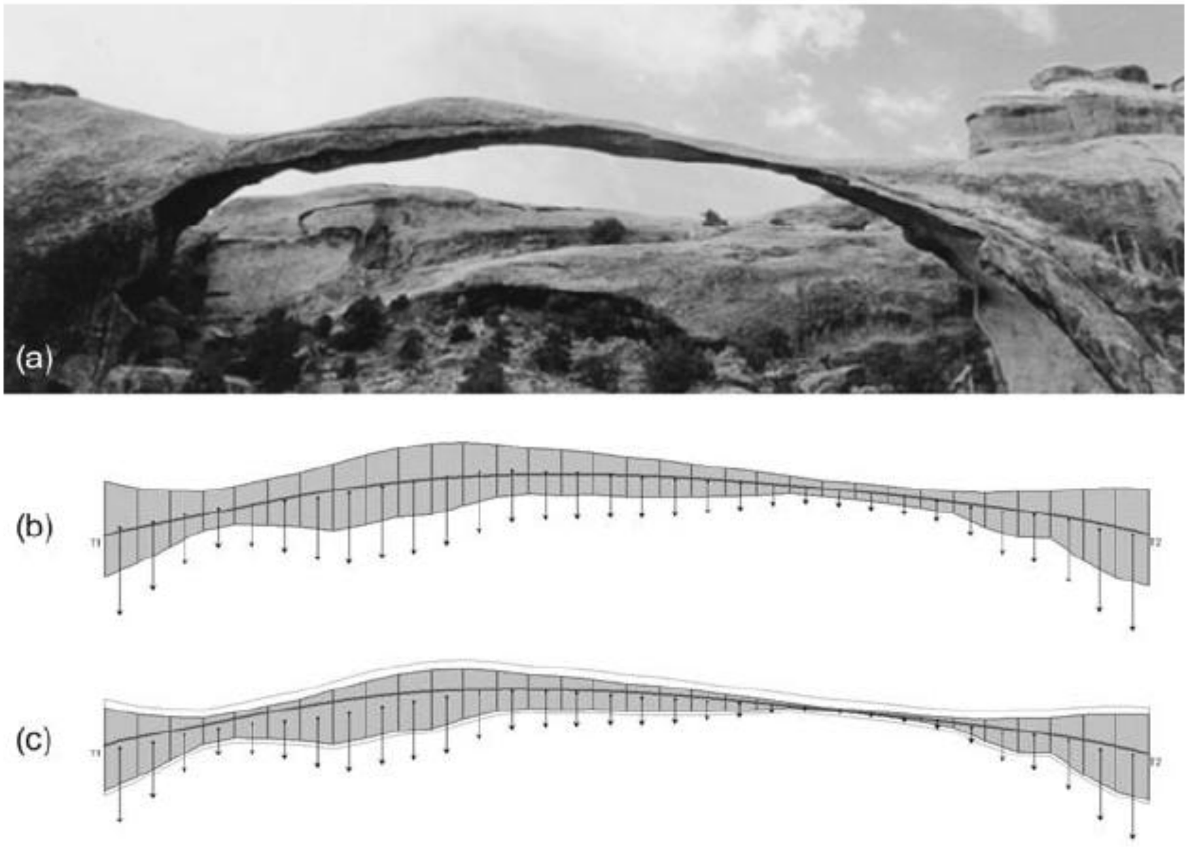



Figure 50 - Block's 'Landscape Arch' (a) and a model showing a possible thrust line of this natural arch, currently (b) and after (c) erosion simulation (Block 2005).

The work by Block (Block et al. 2006), propose structural analysis tools based on the limit state analysis for vaulted masonry buildings. This study extends the graphical method for limit analysis using the line of thrust. Nevertheless it does not provide a complete analysis of three-dimensional behaviour and mechanisms of vaulted structures (D'Ayala & Tomasoni 2008).

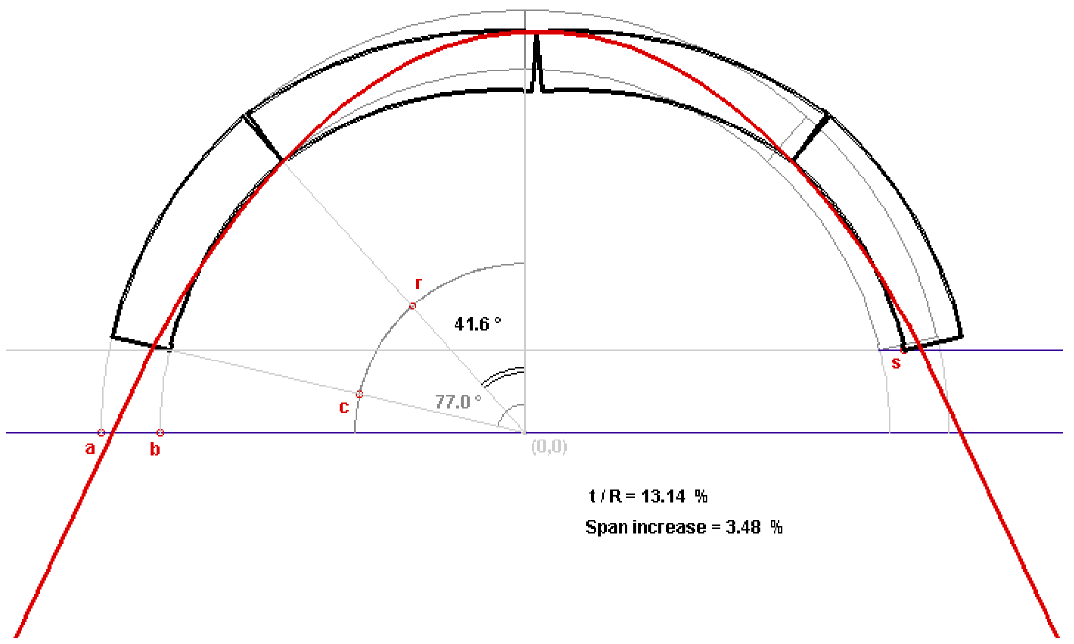



Figure 51 - Interactive thrust applet by Block et al.

## 4.3 Examples of use:

### *4.3.1 The architect Rubió i Bellver graphic static analysis of the Mallorca cathederal (1912):*

In 1912, Joan Rubió studied the Mallorca cathedral using the static method. He used a thrust line analysis in two dimensions and could find a solution for which the thrust line is contained within the thickness of the elements. Safety was ascertained in terms of material strength.

According to (Roca 2001), Rubió was able to find an equilibrated solution for which the thrust line kept fully contained within the volume of the elements. Rubio also admitted that fitting the descending thrust line within the volume of the pier revealed extremely difficult. In his solution, the thrust line becomes almost tangent to the perimeter of the pier at the level of the springing of the lateral vault (figure 52-53). Rubió noted that this solution was consistent with the curvature shown by the pier.

Rubió was not fully satisfied with the solution obtained because it produced a very demanding, extreme condition in some of the structural elements and, in special, in the piers. However, all his attempts to find an alternate, less demanding form of equilibrium failed; this fact, still, does not mean that his solution is the only possible. As stated by himself, "*the solution obtained, even if satisfactory, does not fully content the spirit nor it is unquestionable*" (Roca 2001).

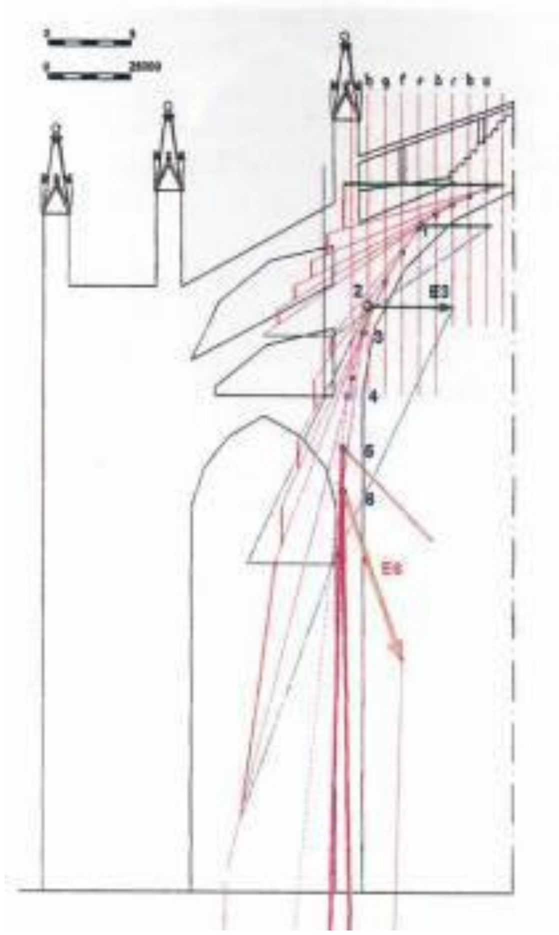



Figure 52 Equilibrium analysis of the Mallorca cathedral

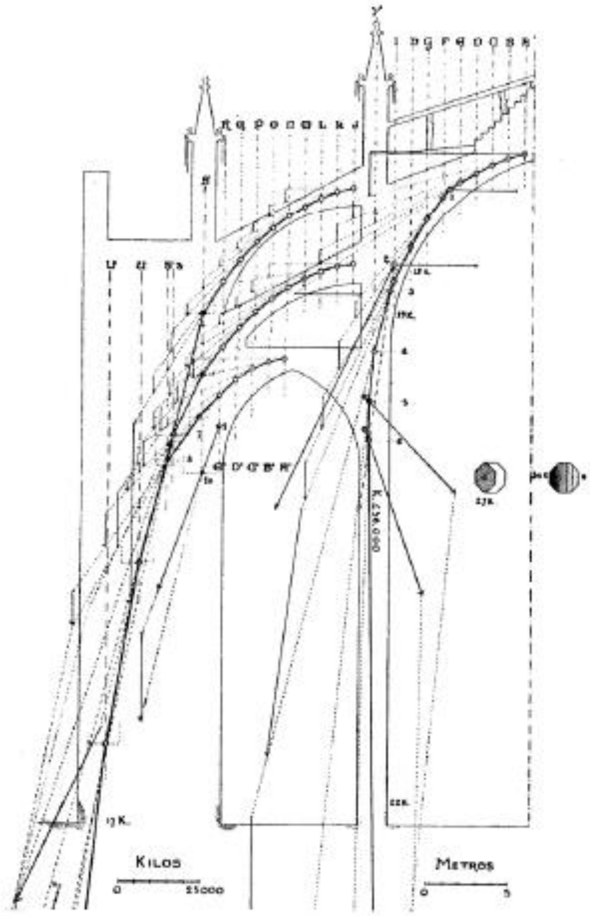


Figure 53 Rubió study (Das, 2008)

### 4.3.2 Analysis of the structure of the church of Sankt Martin in Landshut by Zorn:

Zorn (1933) analysed the structure of the church of Sankt Martin in Landshut. In this analysis, Zorn was trying to understand how the builders of the church achieved an equilibrium status for the slender vertical pillar (Huerta 2001). The lateral vault was much smaller that the central vault and the resultant thrust would be unbalanced. The conducted graphical analysis revealed that above the transverse lateral arch there is a thick heavy wall of solid ashlar masonry and at its top charges part of the roof. The builders of the church increased the load of the lateral transverse arch so as to augment its thrust and, finally, be able to equilibrate, almost exactly, the thrust of the central nave. The analysis showed that the load went down almost vertically through the pillar. In this case, the proceeded graphical analysis demonstrated the balance technique used by the masons to achieve the equilibrium.

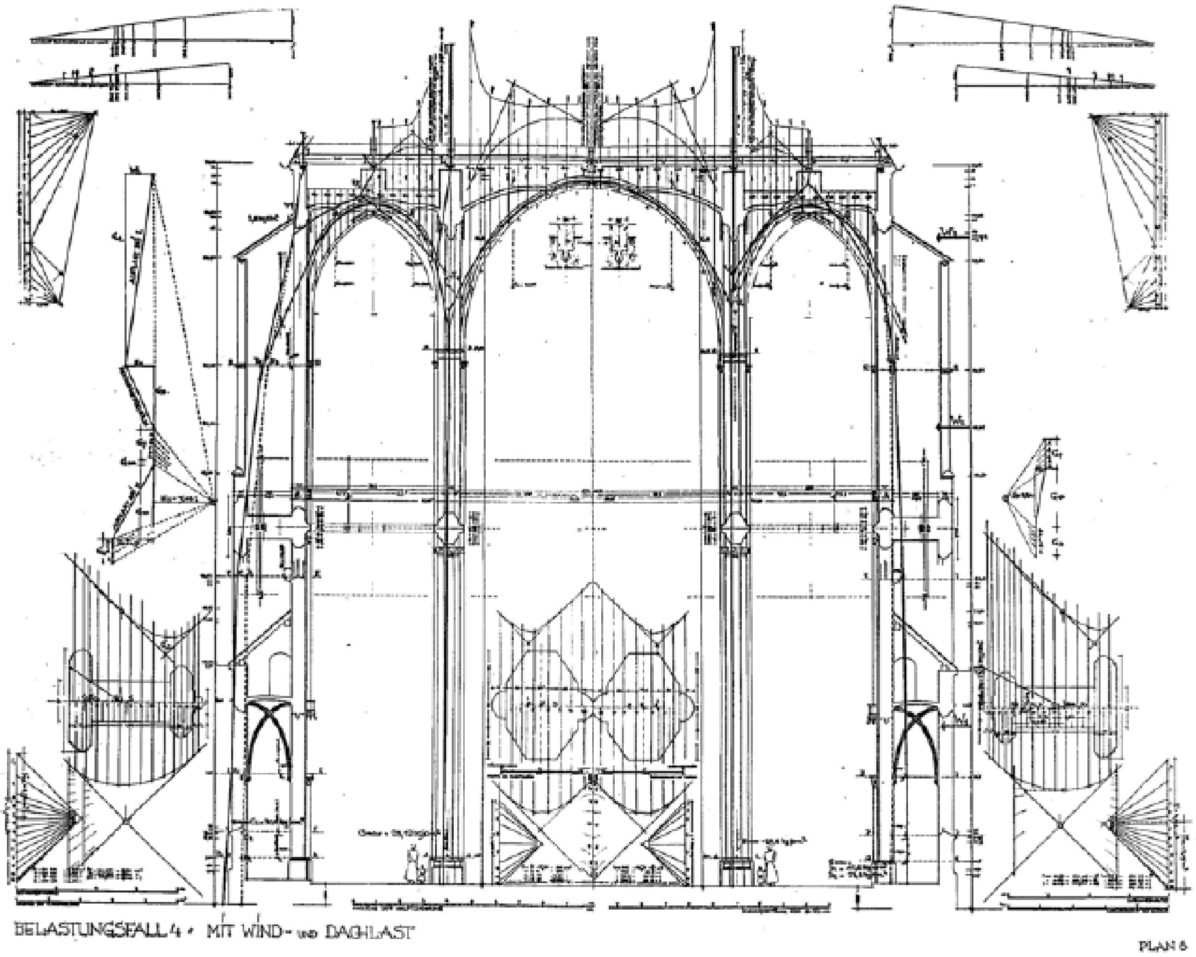



Figure 54 Equilibrium analysis of the church of Sankt Martin in Landshut (Zorn 1933)

### *4.3.3 William S. Wolfe's graphical analysis:*

Based on the membrane theory of domes, Wolfe's method assumes internal forces act along an imaginary membrane with zero thickness at the median radius of the dome section. Similar to the analytical approach to the membrane theory, this graphical method provides only an approximation of internal forces. His method approximates the dome, or vault, as a number of slices or lunes in which two opposing lunes form a quasi two-dimensional arch (Heyman 1977).

In his study, Wolfe using the method of slicing and using approximation, he studied a square-bayed rib vault. Actually, he succeeded to find a line of thrust passing by the middle of the rib. However, it should be clarified that he took an assumption regarding the loads applied on the rib as the lines of pressure must be horizontal at the center (Wolfe 1921).

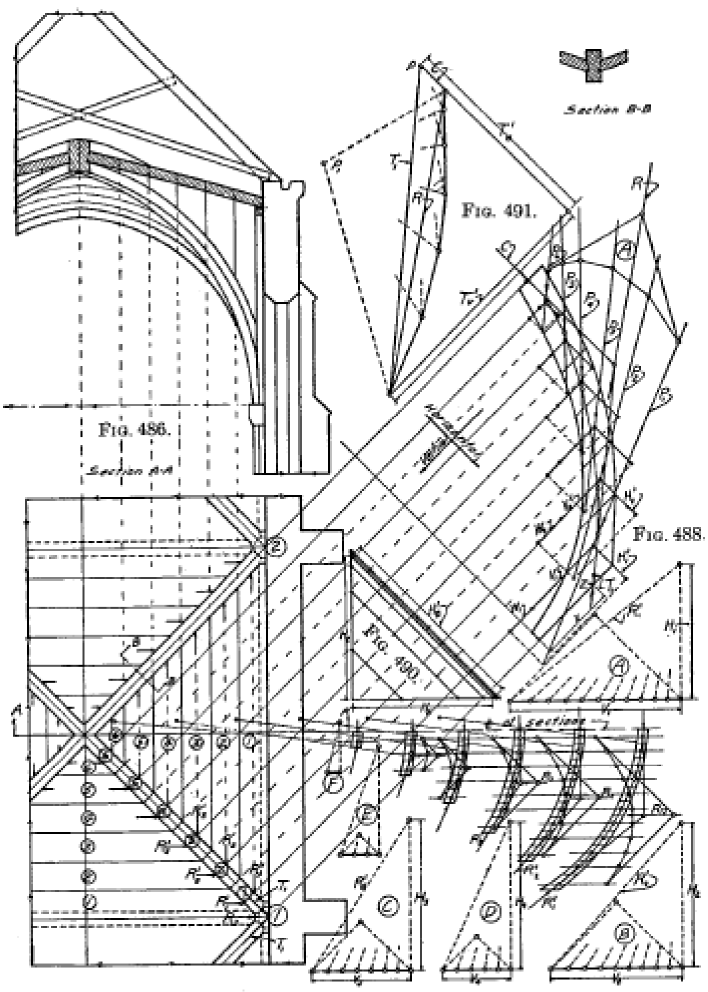



FIG. 487. FIG. 489.

Figure 55 Wolfe’s graphical analysis of a square-bayed rib vault, by the equilibrium method (Wolfe 1921).

## 4.4 Graphical analysis software and applications:

### *4.4.1 Ring:*

Ring is a commercial software to implement the rigid block limit analysis technique. By using rigorous mathematical optimization solvers, the software directly identifies a wide range of potential modes of response. [3]This includes the following;

- Hinging and/or masonry material failure
- Radial sliding failure between voussoirs
- Slippage between rings or ring separation

[3] Ring, Theory & Modeling Guide, version 2.0j

- Identification of the critical failure mode for multi-span bridges with stocky or slender intermediate piers whether this involves one, two or more spans
- Movement of the supports

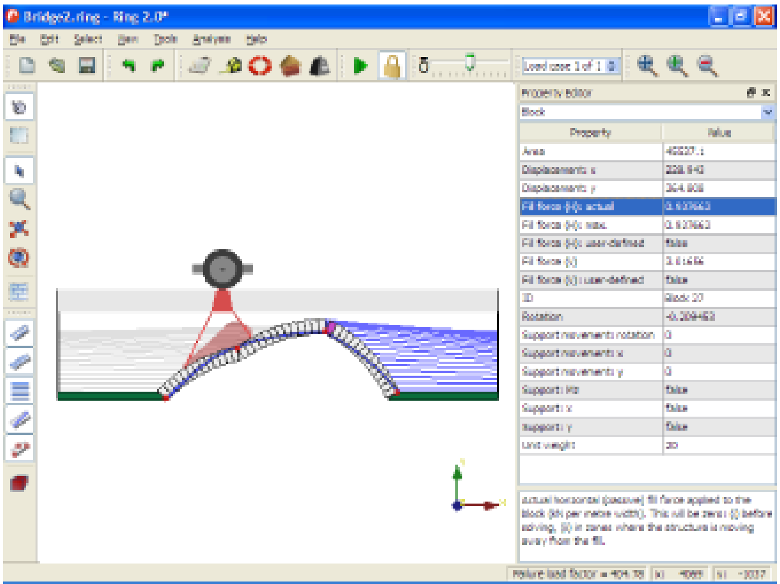



Figure 56 - the Ring software (Ring)

### 4.4.2 *ArchieM:*

Archie-M is a Windows program designed to help with the analysis of masonry bridges and viaducts. It is based on the principle of the thrust line, and provides rapid, interactive equilibrium analysis of arching structures. It is designed to help with the analysis of arch bridges and viaducts as part of an assessment or design process (ArchieM)

The program uses the zone of thrust to define the minimum structure necessary to support a given set of loads. Graphical and numerical results are updated continuously as loads are moved across the structure by hand or automatically.

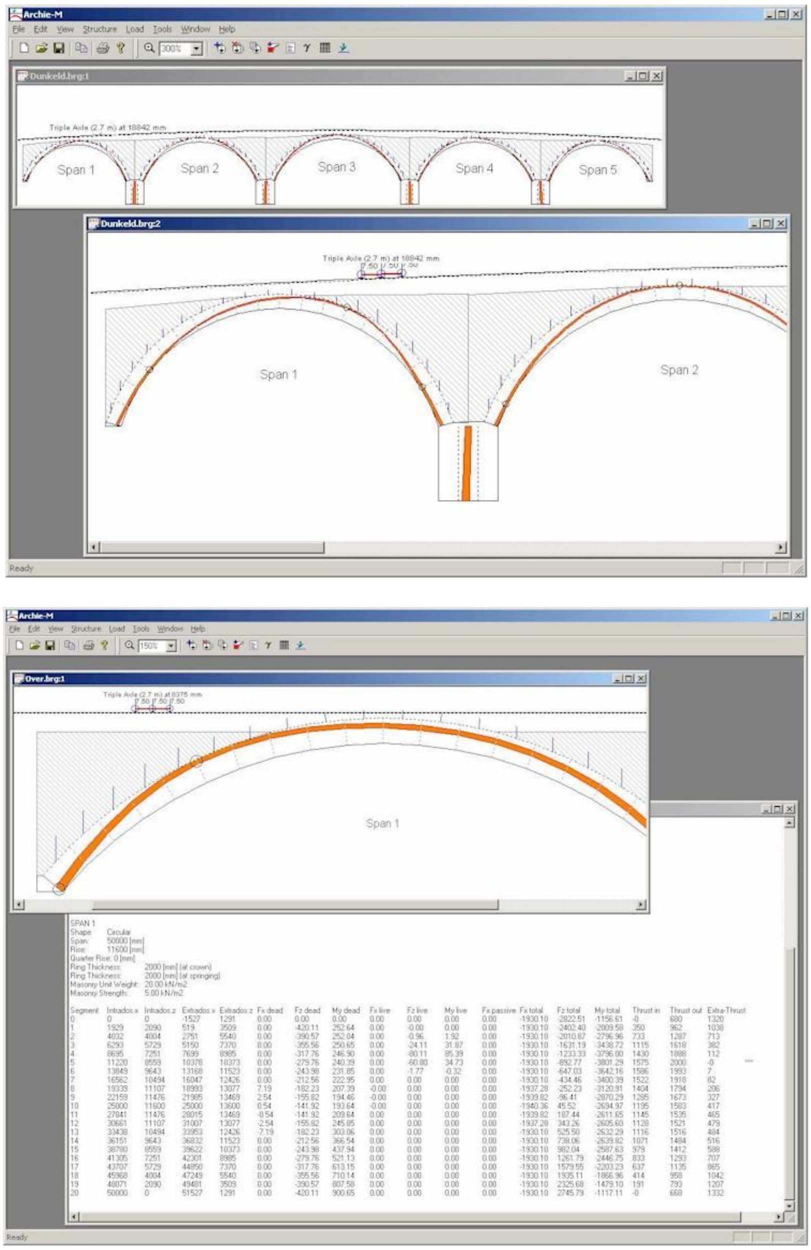



Figure - 57 the use of software (ArchieM).

### 4.4.3 Cadenary:

Cadenary is a digital hanging chain model based on a particle spring systems using Euler and Runga Kutta solvers programmed in Java. Its goal is to provide a real-time 3D-modeling environment that allows the design of gravity based forms following the hanging chain principle (Kilian 2005).

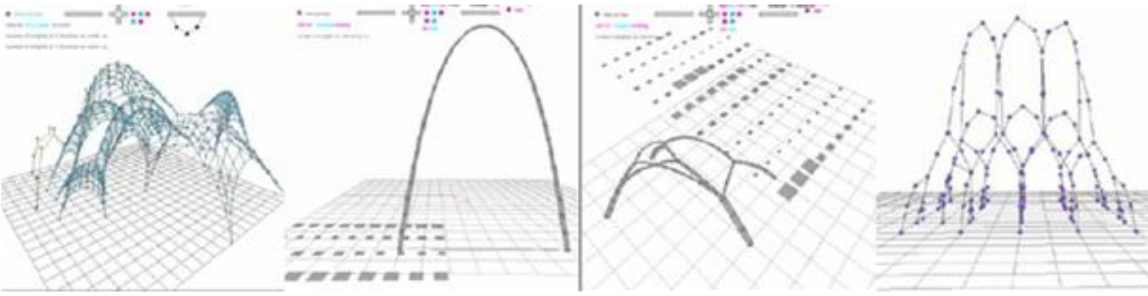


Figure 58 - Cadenary tool (Kilian 2005)

## 4.5 Conclusions:

- Different analysis method were developed in order to understand or to analyse the masonry structures and the gothic vaults.
- The analysis of historical masonry structures is a complex task and only few proposals specifically oriented to the non-linear analysis of masonry domes and vaults exist.
- Most of the analyses techniques neglect the three-dimensional behaviour and mechanisms of vaulted structures meaning they do not take into account the interaction between the arches and it is not able to properly simulate the three-dimensional effects in the vaults.
- Graphical (Thrust line) analysis is a simple and fast method to determine the failure of the arches, and can be applied to the gothic vault by following a simplification process as using the slicing technique.
- The graphical analysis, being a relatively traditional technique to the FEM modeling, is still in use and also subjected to computerized development such as the Excel spreadsheets and the analysis software such as Ring or ArchieM.
- The previous proceeded analyses demonstrate that the gothic masters were controlling the equilibrium of the structure playing with the loads as in a balance while following strict geometrical rules.

# 5. CASE STUDIES

## 5.1 The approach:

The behaviour of the cross vaults, specially the gothic vaults, was always a subjected to debates. Mainly two main theories regarding the load paths and the thrust allocation within the statically stable parts of the vaults were developed. The first one assumes that the load paths are acting parallel to each other and following the arches of the vaults, as can be seen in figure 59. The other theory disagrees with the previous and assumes that the load paths are not parallel and they act in diagonal arches which convert at almost one point located at the lower point of the vault haunches (figure 60).

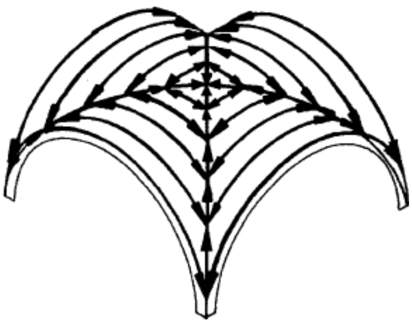


Figure 59 - A possible load path for a groined vault (O'Dwyer 1999).

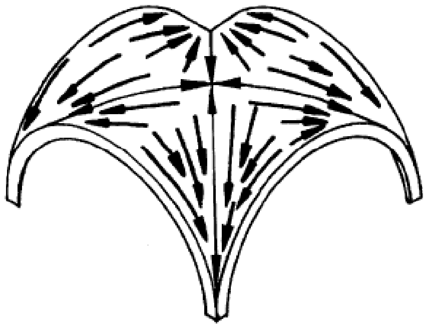


Figure 60 - An alternative load path for a groined vault (O'Dwyer 1999).

Moreover, the combination of the previously mentioned theories was proposed by different researchers. This implies that the two intersected vaults will behave in two different ways as one vault will have a parallel load path and the other vault will have a diagonal load path. This combination was mainly suggested for the quadripartite vault with rectangular projection. However, none of these theories for the gothic vaults were analytically approved or proven to be wrong.

The approach of this study is to analyse a gothic intersected ribbed vault following these theories, to compare between these theories and to evaluate the results. For this purpose, two historical gothic structures located in Barcelona, Catalunya and dating to the late 14$^{th}$ century were chosen for this study. It is to be mentioned that dealing with the gothic vaulted structures, and generally the historical, imply many difficulties due to the lack of information about the materials used, the hidden geometry of many elements such as the filling and also its dimensions. Generally, these types of structures are quiet ambiguous. Also, the use of non destructive and minor destructive testing does not provide all the information needed for the analysis. Thus, in different cases, I had to take some assumptions that will be explained during the presentation of the analysis.

It is important to underline that during the analyses presented; I took into consideration the main aspects of the modern theory of Limit Analysis of Masonry structures developed by Professor Heyman. They imply the safe theorem assumptions of infinite compressive strength, friction resistance and zero tensile strength for the stone masonry.

In order to analyse the vault, the following steps for the used technique were applied:

The geometry of the vault:

> First step was the analysis of the geometry of the vault, which implied a deep understanding of the structure and its different elements. This was done by visiting the actual site, taking pictures and by reading the available and provided drawings and documents.

The different elements and their properties:

> The second step was that I defined the sub elements of the intersected vault such as the rib, the longitudinal vault, the transversal, the 2 kinds of infill, the transversal arch and all its other subcomponents. It was very important to define the dimensions of each component, its material and the relative density for each of them.

Modeling the vault, discretization and cross sections technique:

> Drawing the different cross section one by one was not for me a realistic approach, due to the complexity of the vault and the large needed number of sections. I used to model the vault using the Google Sketchup software and to discretize all the different components of the vault. This software allowed me to define section plans at exactly the location for the cross sections I needed for my analysis.

Development of excel spreadsheet for the limit analysis of sections

> Due to the number of sections required for the analysis, the graphical analysis method using the cad software was time consuming and unfeasible. Thus I used Microsoft excel software in order to design a spreadsheet for the graphical limit analysis. Once again, for the discretization of the cross section in excel, I used a special macro to export point by point the true shape of the arches from the CAD drawings into excel spreadsheets.

Analysis of the different sections.

> Finally, I entered all the data related to each cross section in individual Excel sheets; these data included the different coordinates of the section’s geometry and its dimensions. Also, it included the relative densities for each material. The spreadsheet is designed to calculate the different weights for the sub divided voussoirs of the arch and of the cross section. And by applying and changing an assumed value for a lateral force, I get a line of thrust for each arch.

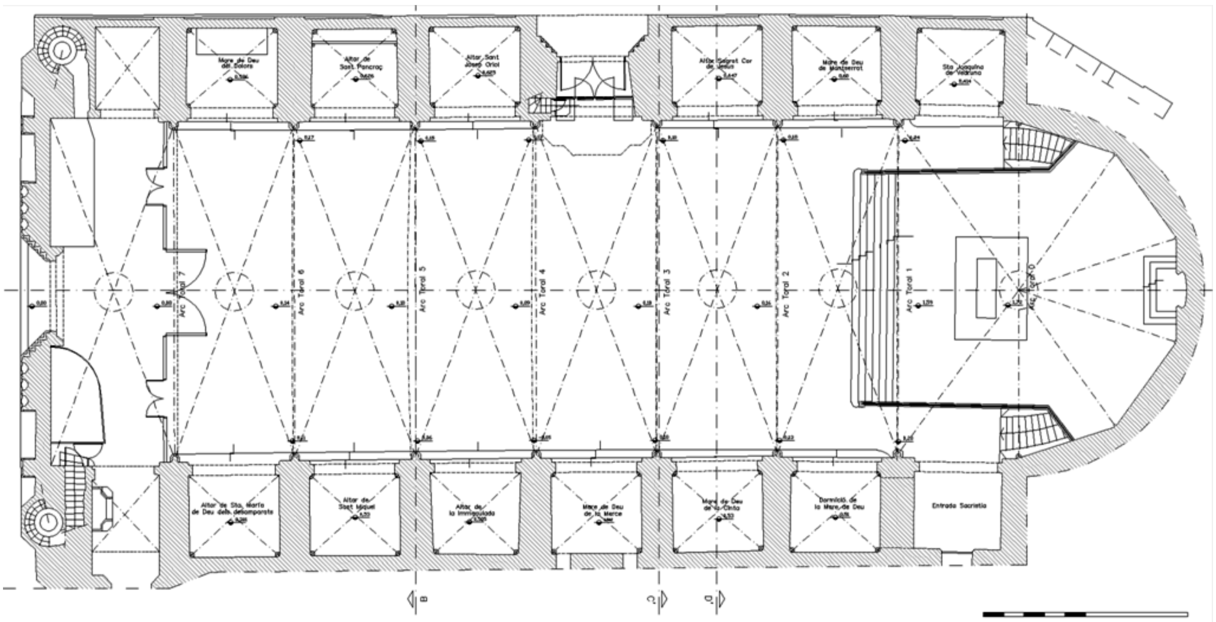
In the following section, the steps of analysis will be explained in details and accompanied by drawings, sketches and graphs.

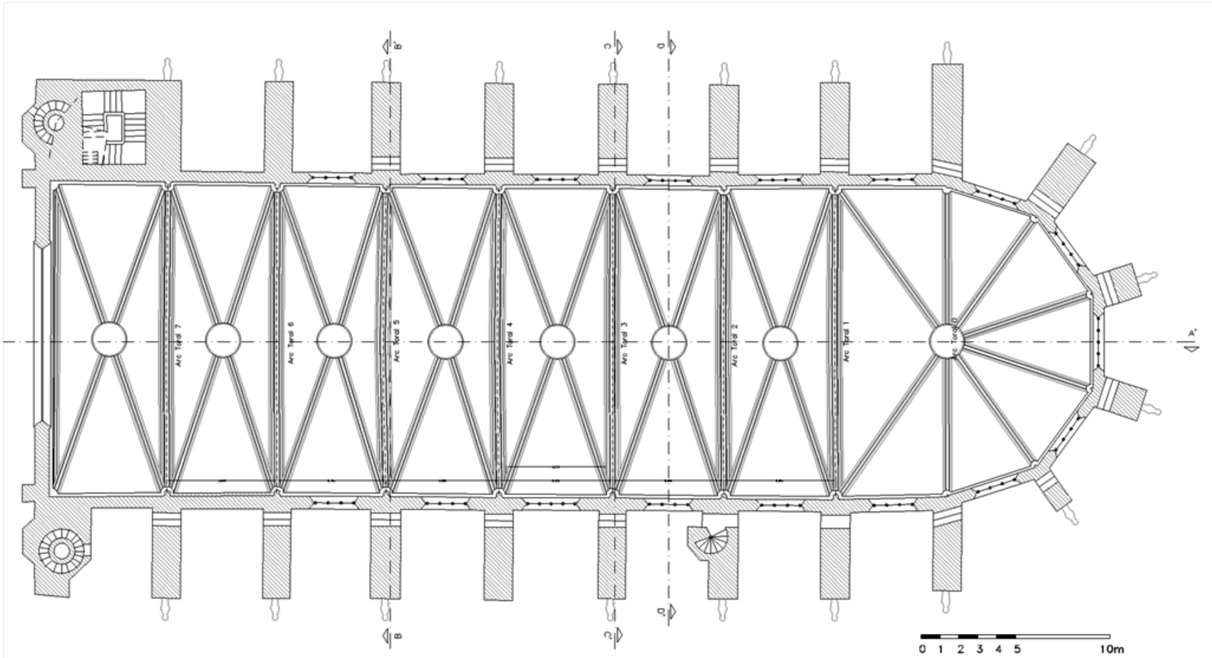
## 5.2 Santa Maria Del Pi Church:

### *5.2.1 General description of the church:*

The church of “Santa Maria del Pi” is located in the ancient district of Barcelona (Spain), known today as the “Pi neighborhood”, a neighborhood that was settled during c. XIII by the construction of the new city wall. This monument is composed by structures of different chronologies among which; the church itself, the bell tower, the chapel of “Sang” and the rectorate which are dated from the early medieval period.

Santa Maria del Pi, being a religious building from the latest gothic (the construction of the church is known to be finished by the end of the 15th century), has a classical basilica plan with a single nave of 55m length, 16m width and 27m height. It has lateral chapels between its buttresses and a polygonal apse on the southwest where two more chapels are located at each side of the presbytery, a sacristy and a main stair to enter the bell tower and the roof. The nave presents an internal organization of seven rectangular sections of approximately 6m each one covered with quadripartite vaults; whereas the apse and the presbytery are covered with a palm-shaped ogive vault. The toral and ogival arches of the roof of each section of the nave rest on the columns inside the buttresses separating the chapels. From the level of the impost these capitals form the cornice that crosses the whole perimeter of the church and from which the windows of the cloister go out.

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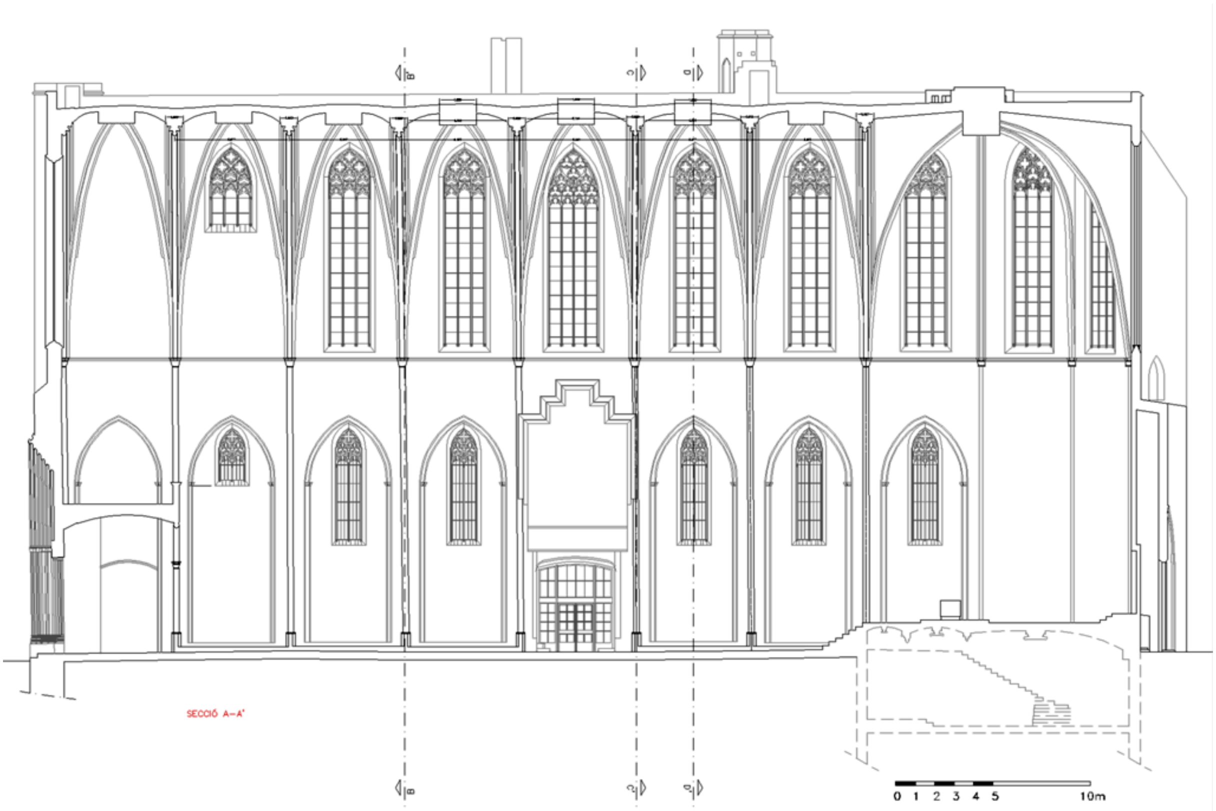
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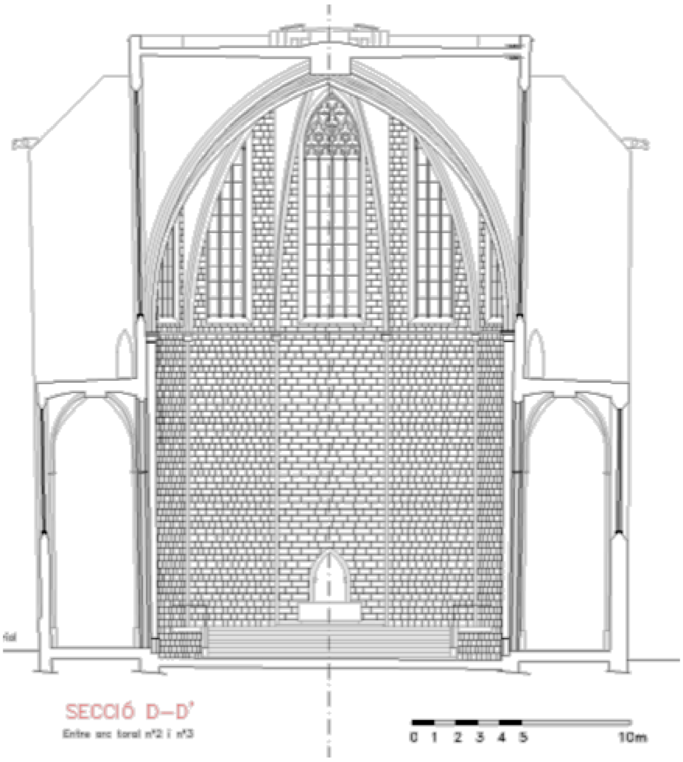



Figure 61 Architectural drawings of the church (plans, Long. and Trans. sections).

Figure 62 Interior of the church of SM del Pi

The stone used for the construction of the church is sourced from the “Montjuic” quarries, corresponding to a silicic stoneware which characteristics have a high durability and resistance.

Generally, the stones forming this religious building are of good quality in composition and granulometry. The mortar is lime based of fine granulometry and flattened in order to limit the entrance of water.

### 5.2.2 The Geometry of the Vault:

The SM del Pi vault can be defined as a repetitive roofing structural element which covers the main nave of the church and each vault is bordered by the longitudinal walls and buttresses of the church from one part and by the repetitive transversal nave arch from the other part. It's to be noted that it represents a typical Catalan vault which is known for its slenderness and the use of relatively thin stone for its construction. The vault consists of two intersecting and perpendicular pointed vaults; transversal and longitudinal having a rectangular plan projection of 16.57 by 5.26 m. They are constructed using the previously mentioned "Montjuic" stone. Underneath, there are two intersected cross ribs and are supporting a considerable mass of stone which is the keystone. The cross ribs start from the point of intersection of the transversal nave arch with the walls.

Figure 63 Picture showing the repetitive cross vaults

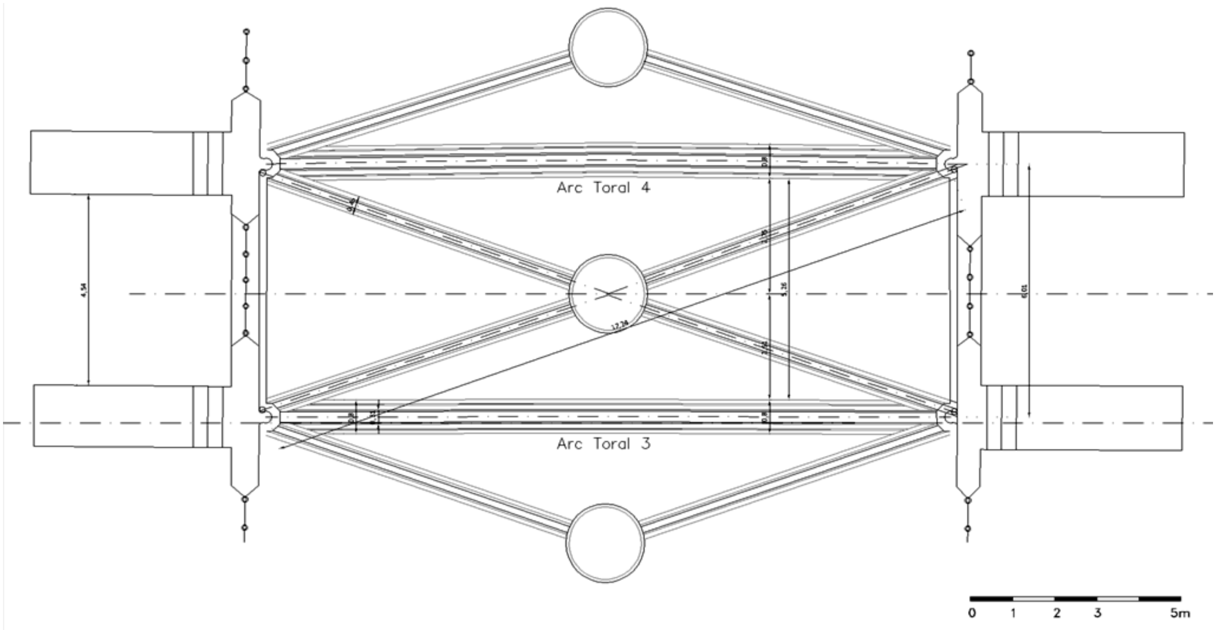



Figure 64 Plan showing the repetitive intersected vault

Above the stone membrane, there exist different layers of infill and roofing materials, varying in density and in mechanical properties. The highly dense material, consisting of mortar and stone, is located at the lower levels till the haunches and will be called sound infill. The upper levels contain the lower dense materials, consisting of rubble, and will be called light infill. On the top, there is the tiling pavement which has slops for the rain circulation and drainage.

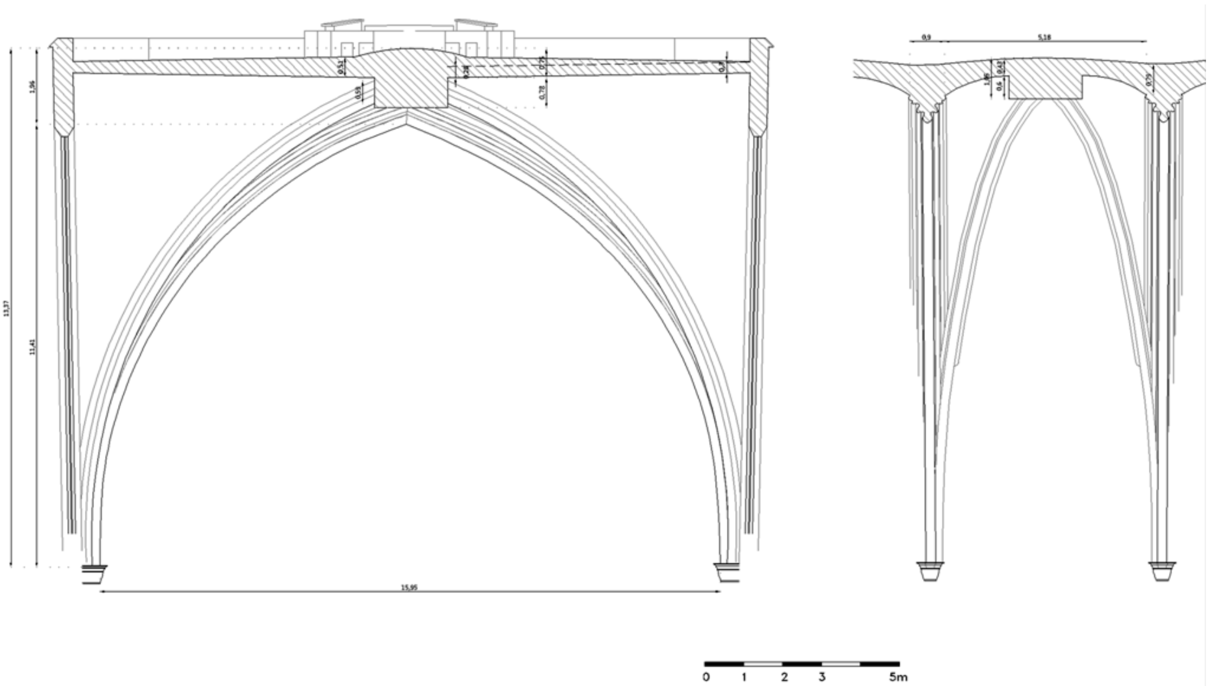



Figure 65 Longitudinal and transversal sections of the vault

## 5.2.3 The different elements and their properties:

### 5.2.3.1 The transversal arch:

This pointed arch is the main arch of the nave and is supported from its both sides by the buttresses. It consists of multiple voussoirs of the Montjuic stones and the stone, including the mortar joints, is given a mass density (specific weight) equal to $\gamma$=22 KN/m$^3$. Its dimensions and the shape of the cross section were identified from the drawings. The arch had a free span of 15.75m and a rise of almost 10m. The cross section has an ornamental triangular shape with 0.89m width and 0.71 m thickness as shown in figure 67. However, the layers directly above couldn't be identified whatever only infill or that some stone layers existed above. I assumed the existence of only the infill with tile pavement on the top.

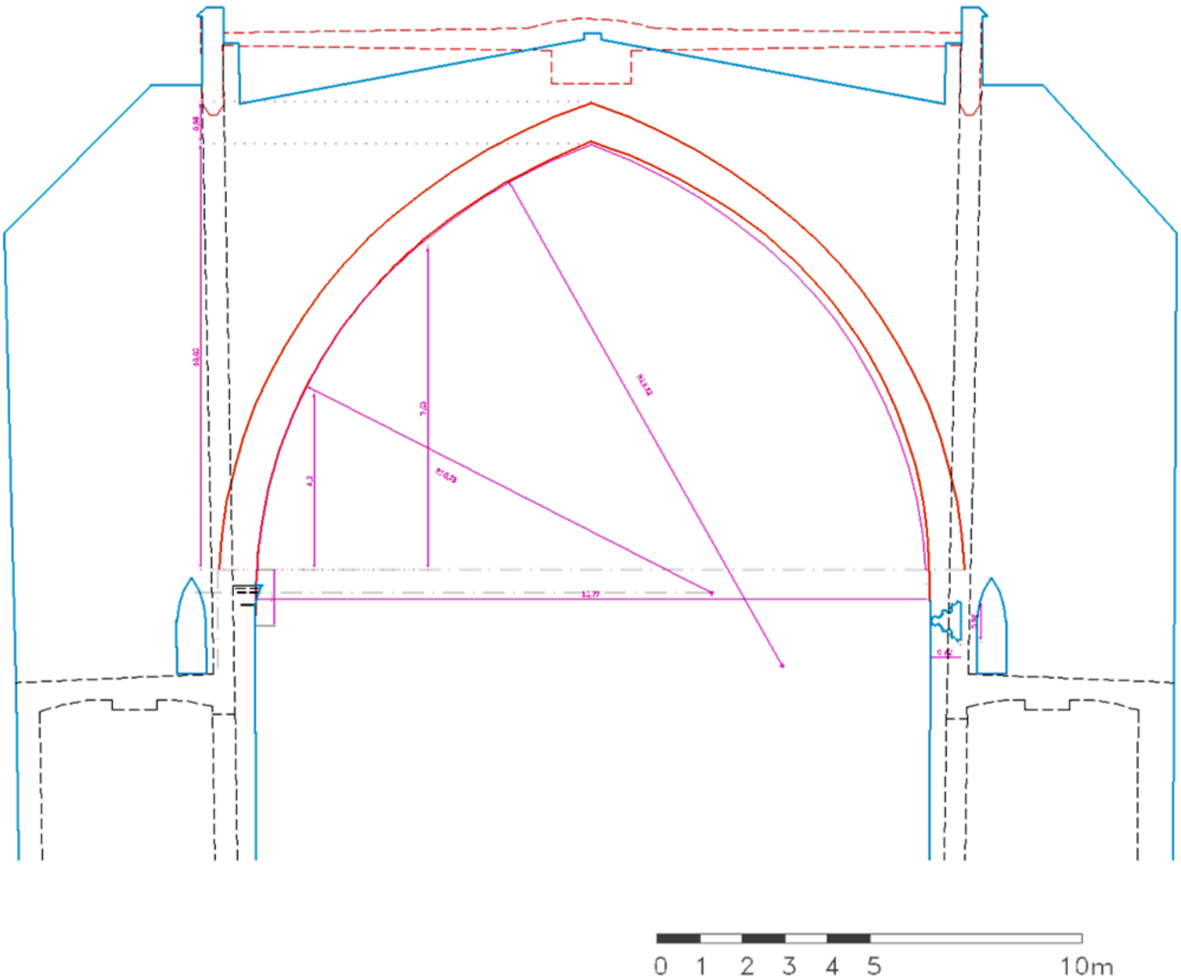



Figure 66 Geometrical identification of the main transversal arch

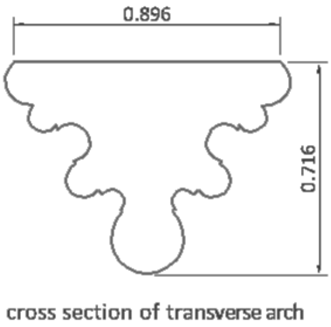



Figure 67 Cross section of the transversal Nave arch.

#### *5.2.3.2 The cross ribs:*

They are formed of multiple voussoirs similar to the main transverse arch, but in smaller size. Also following a pointed arch shape they traveled upward from the intersection between the side walls with the transversal nave arch and they converged with a huge block of stone (the keystone) at the middle of the span. They had a span of 17.24 m and a rise of almost 11m and the stone specific weight is equal to 22 KN/m$^3$.The dimensions for the geometry of their pointed arch profile were not available in the provided drawings of the church; only a lateral projection in the transversal section. Thus, I derived their geometry by following a point by point projection form the plan and the side view elevation as shown in figure 68.

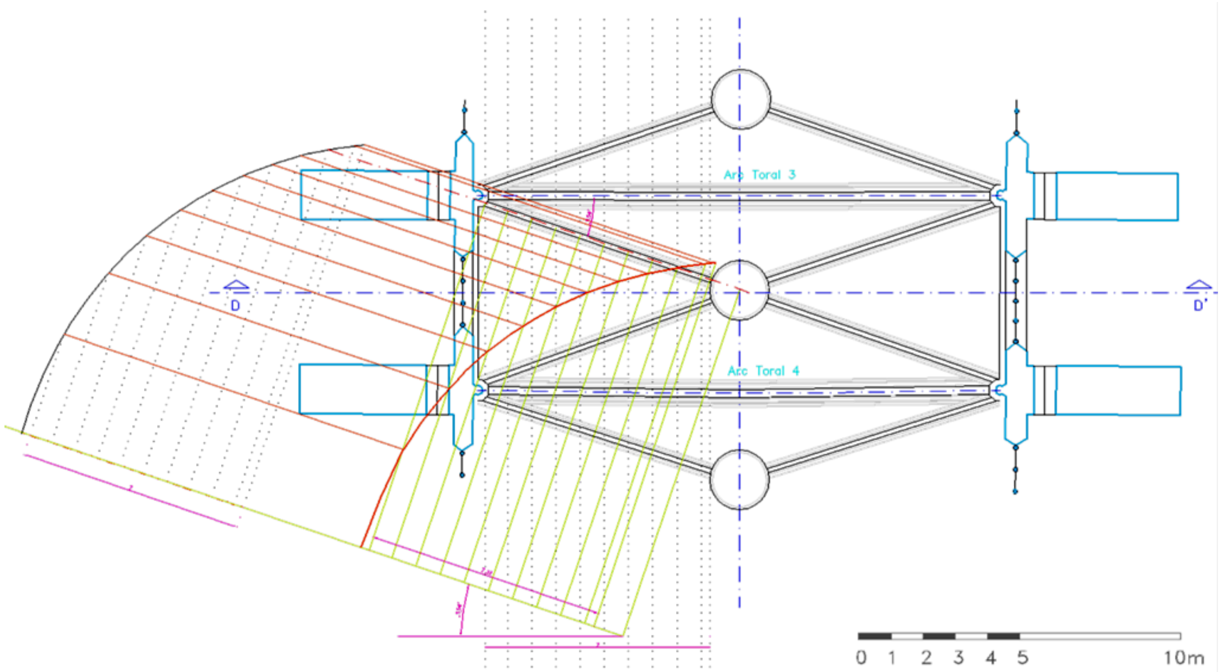



Figure 68 The derived geometry of the cross rib (highlighted in red)

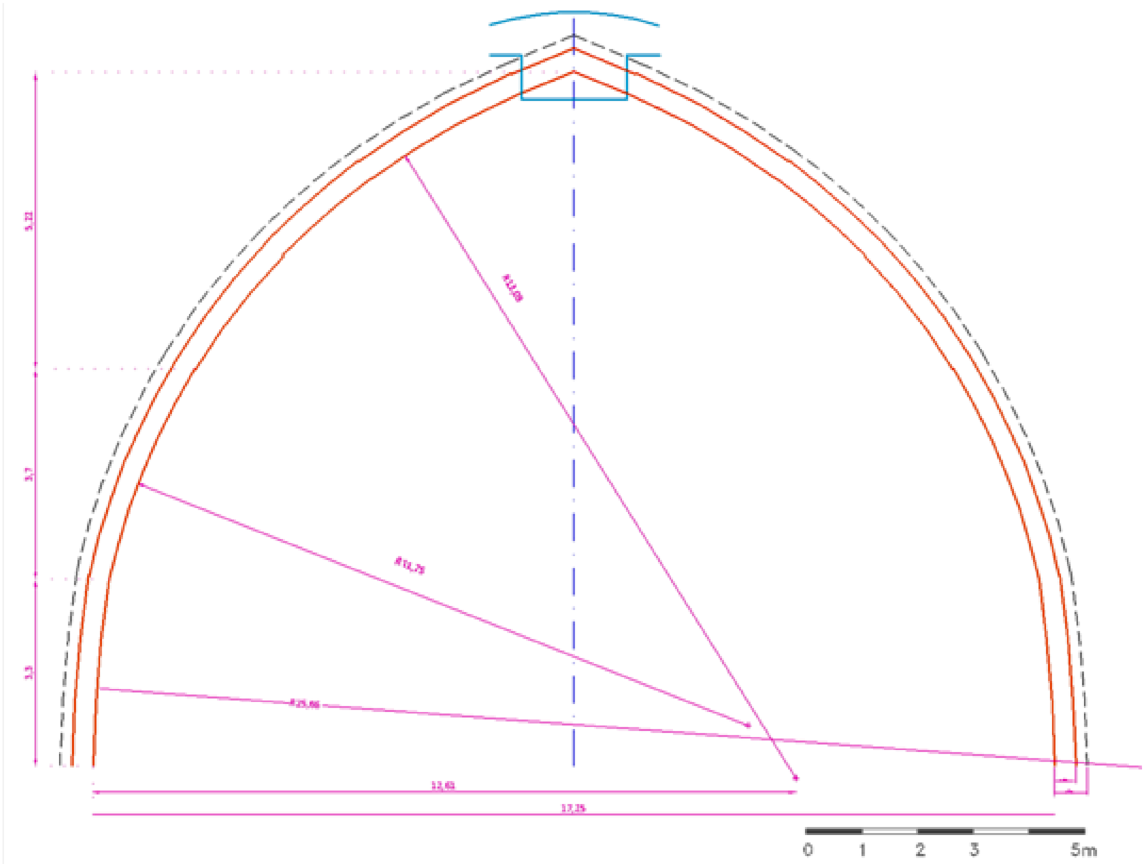



Figure 69 derived geometry of the cross rib

Also, the cross section of the cross rib was not given in any of the provided drawings and thus I had to assume the shape and dimensions of the section. From the visual investigation of the vault in site (as can be seen in figure 70, the shape seemed quite similar to the one of the transverse arch but in smaller dimension. Based on that, I derived the geometry as shown in figure 71.

Figure 70 Picture showing the transverse arch and cross rib shapes

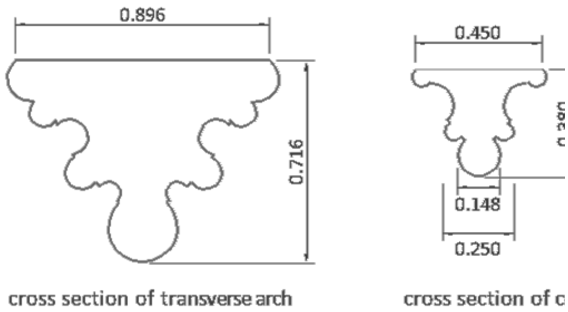



Figure 71 The supposed shape and dimensions of the cross rib section

The given width is 0.45m at the top and 0.148 m at the bottom and the height is 0.38m.

#### 5.2.3.3 *The transversal vault:*

The transversal vault is the part enclosed by the external longitudinal wall and the cross ribs. It is mainly constructed of relatively thin stone blocks around 20 cm of thickness and also follows a pointed arch section shape. The given mass density for the stone is equal to 22 $KN/m^3$. The largest width of the vault is 5.26m and has a rise of almost 9 m.

Figure 72 the transversal vault

Defining the geometry of such a vault was quite problematic as the provided drawings didn't include any cross section to this element especially at the intersection with the wall. I had to derive the profile defining the intrados arch of the vault. I used the projection of the cross rib from the longitudinal section to get this needed profile.

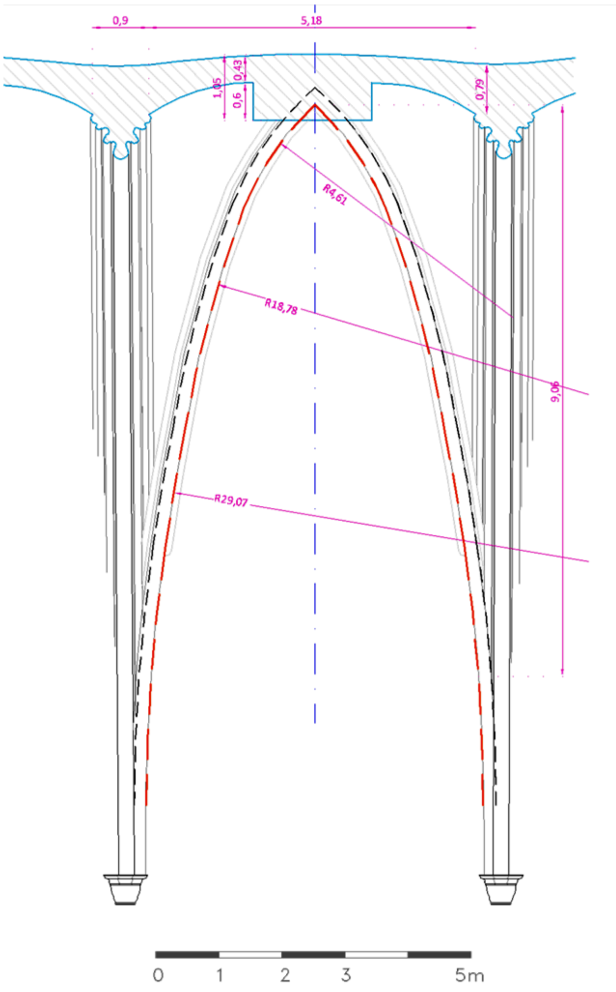
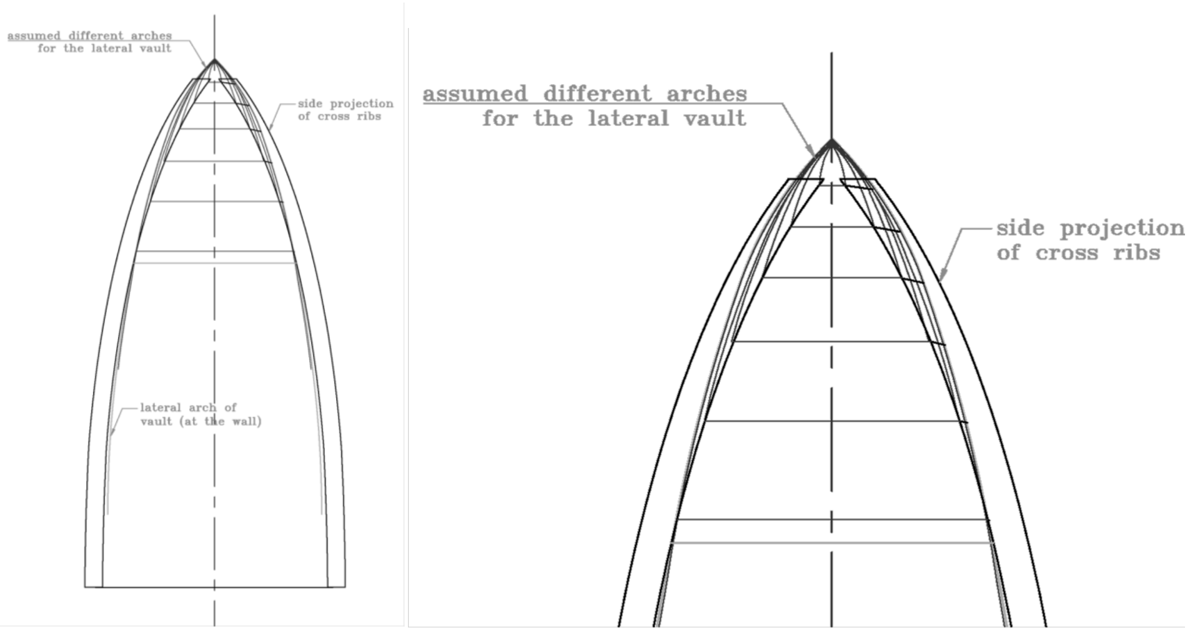



Figure 73 Derived geometry of the transversal vault arch profile

Another geometrical problem I had to deal with was that vault was not similar to a barrel vault meaning that it doesn't keep the same arch profile for its whole length. At each cross section a different profile exists and obviously I can't derive them from drawings. This could only be done following a complex on site survey using laser scanning or photogrammetry in order to capture the 3d geometry.

Figure 74 Different cross section profiles for the transversal vault

Due to feasibility and time issues, I choose to assume a series of cross section profiles but based on visual inspection and the actual geometry of the adjacent elements. Mainly, the vault should start with the derived profile at its intersection with the wall and then its width should follow the decreasing distance between the crossing ribs all along the length of the vault, otherwise the vault will surpass and won't be supported by the ribs. The different cross section profiles that I assumed were taken at 1 m interval and they were scaled from the main profile previously defined.



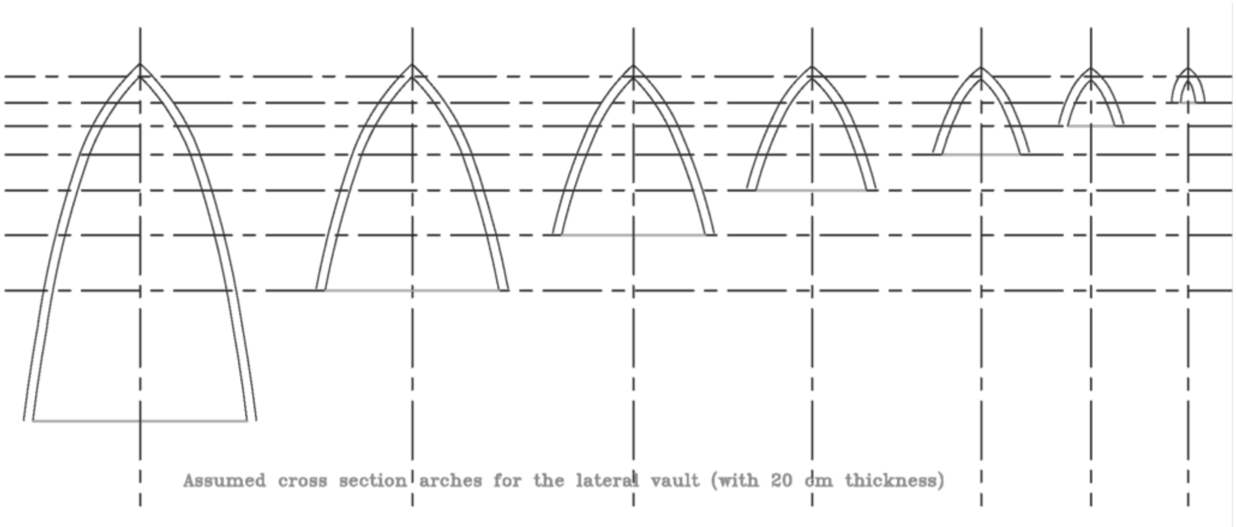



Figure 75 the assumed different arch profiles for the transversal vault

It is to be noted too that the pointed top level of the vault had no constant horizontal level which means there was a slope in the direction of the cross rib keystone equal to 1.5%. Although it is so small, I took it into consideration.

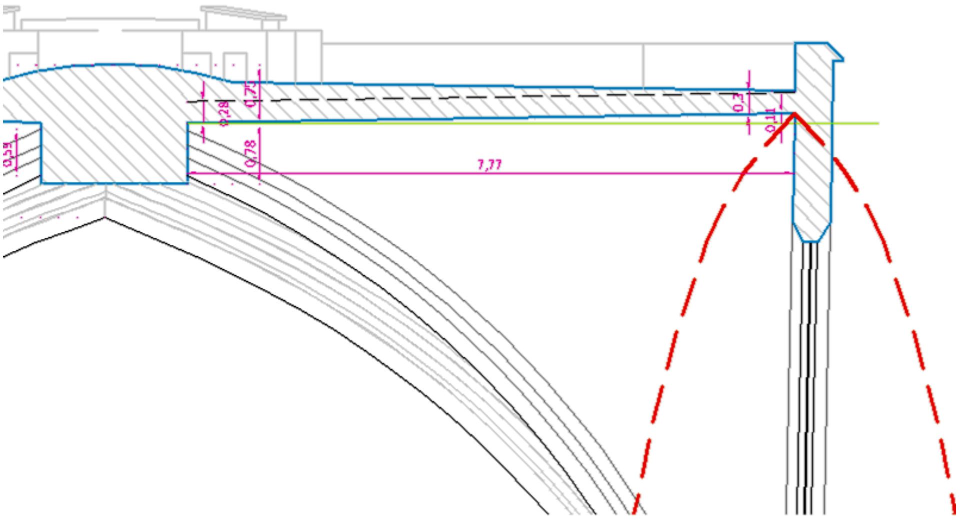



Figure 76 slope of the top level of transversal vault.

#### 5.2.3.4 The Longitudinal vault:

The longitudinal vault is the part flanked by the transversal nave arch and the cross ribs. Similar to the transversal vault, it is constructed of relatively thin stone blocks around 20 cm of thickness and also follows a pointed arch profile shape and is called the web. The given mass density for the stone is equal to 22 KN/m$^3$. The span is 14.58m and the rise is around 5.7m.

Figure 77 The longitudinal vault.

Similar to the transversal vault, the geometry of this web was also a question mark. However, it was more complicated because its shape depended on the two adjacent members; the cross rib and the transversal nave arch. Its geometry has to follow the geometry of both of them. From the visual inspection, I noticed that the vault's web consists of multiple planes following connecting lines, which are the mortar joints that travel from the top of the cross rib to reach the upper profile of the nave arch. This is shown in figure 78.

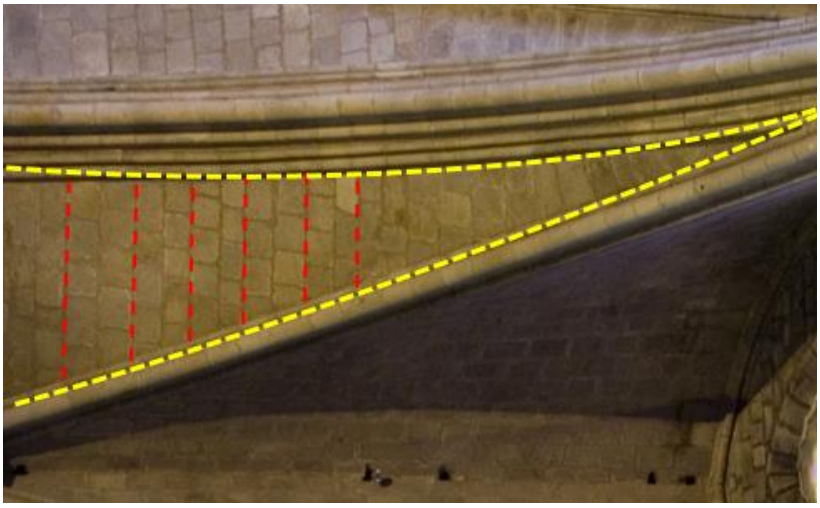


Figure 78 Linear mortar joints in the web.

Moreover, the provided drawings showed that the intrados cross section for the crown of the vault was (the ridge) following a curvature as shown in figure 79. I assumed that this curvature is linear for simplification and that the web is formed of multiple planes.

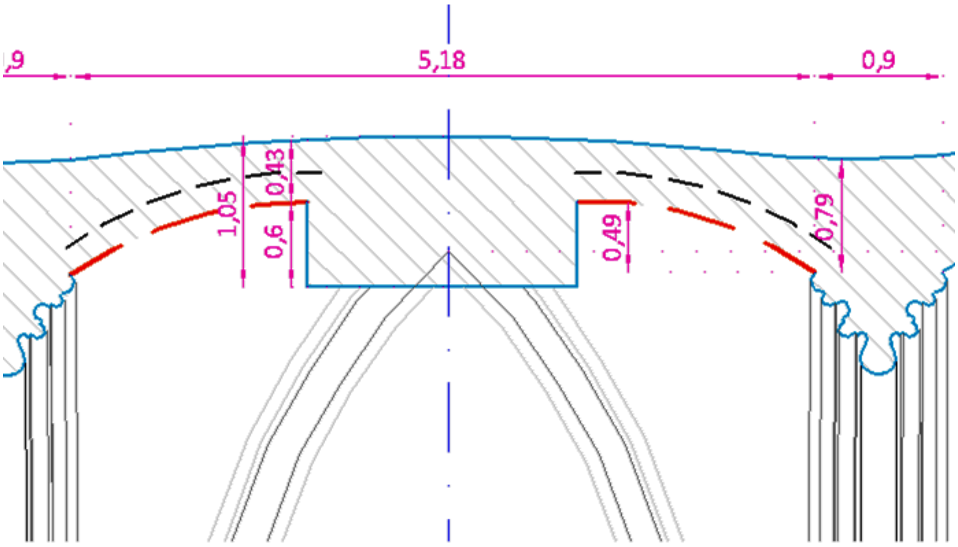



Figure 79 - longitudinal section showing the curvature of the Longitudinal vault.

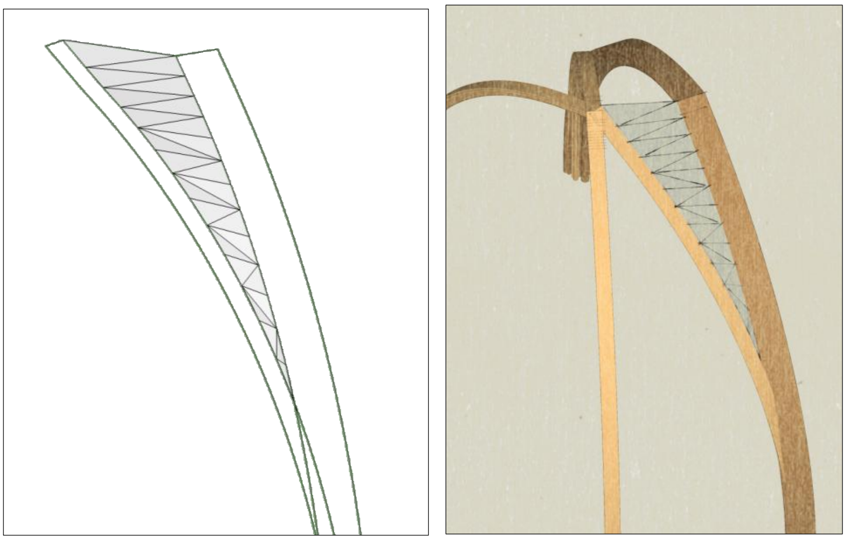


Figure 80 - assumption for the geometry of the web.

#### *5.2.3.5 The Sound infill:*

By definition, the spandrel of the arch is the space between the extrados of an arch and the roadway or the top level of the arch structure. And, this part contains a deposed material called the spandrel filling.[4] Generally two types of material are used for this infill. The first type, which I will call the sound infill, is located at the bottom of the spandrel and rise till 1/3 of the arch height or the equivalent of the ¼ of the arch span. This sound infill is usually a combination of mortar and crushed stone and has a relatively high specific weight. I used the value of 18 KN/m$^3$. It has a structural function to apply some lateral pressure on the arch to contribute to its stability.

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[4] "Cyclopedia Of Architecture, Carpentry, And Building", by James C. et al.

However, to decide the value of the height of the sound infill, there were two alternatives. The first is that the sound infill follows different heights meaning it has a variable height depending on the geometry of the arch it is adjacent to. The second is that it has a constant level for the whole intersected vault. This is demonstrated in (figure 81) by different surfaces with two colors; blue for the first and yellow for the second option.

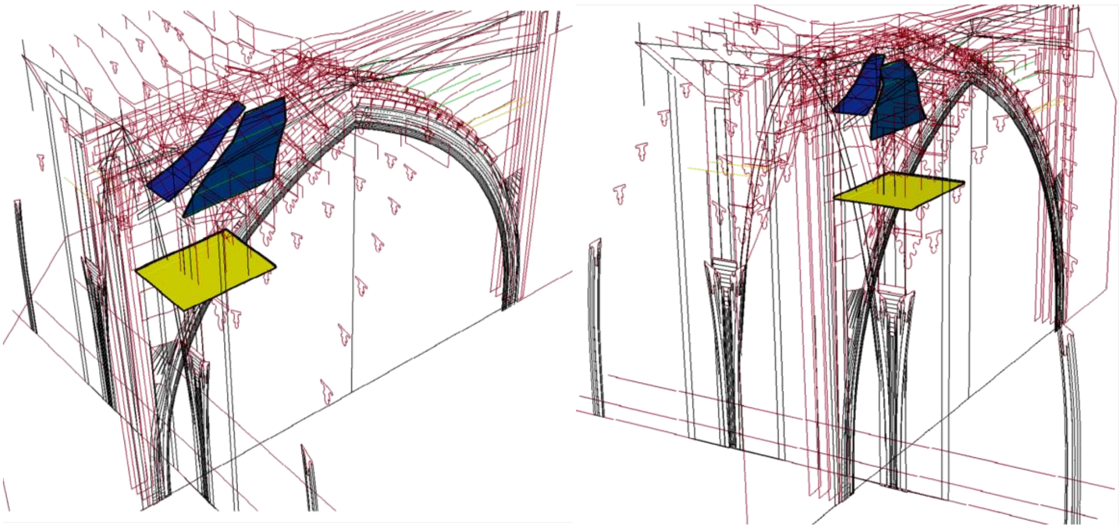


Figure 81 alternatives of changing levels of sound infill (in blue) and a constant level (in yellow)

I assumed that the sound fill should keep a constant height for the whole cross vault (the 2$^{nd}$ option). This can be clarified as the sound infill cannot have an inclined plane and should keep one horizontal level. It cannot start from a lower point and rises to another higher point till reaching the keystone. I choose the lateral transverse nave arch geometry to take the ¼ of the span for defining the height of the sound infill as shown in (figure 82).

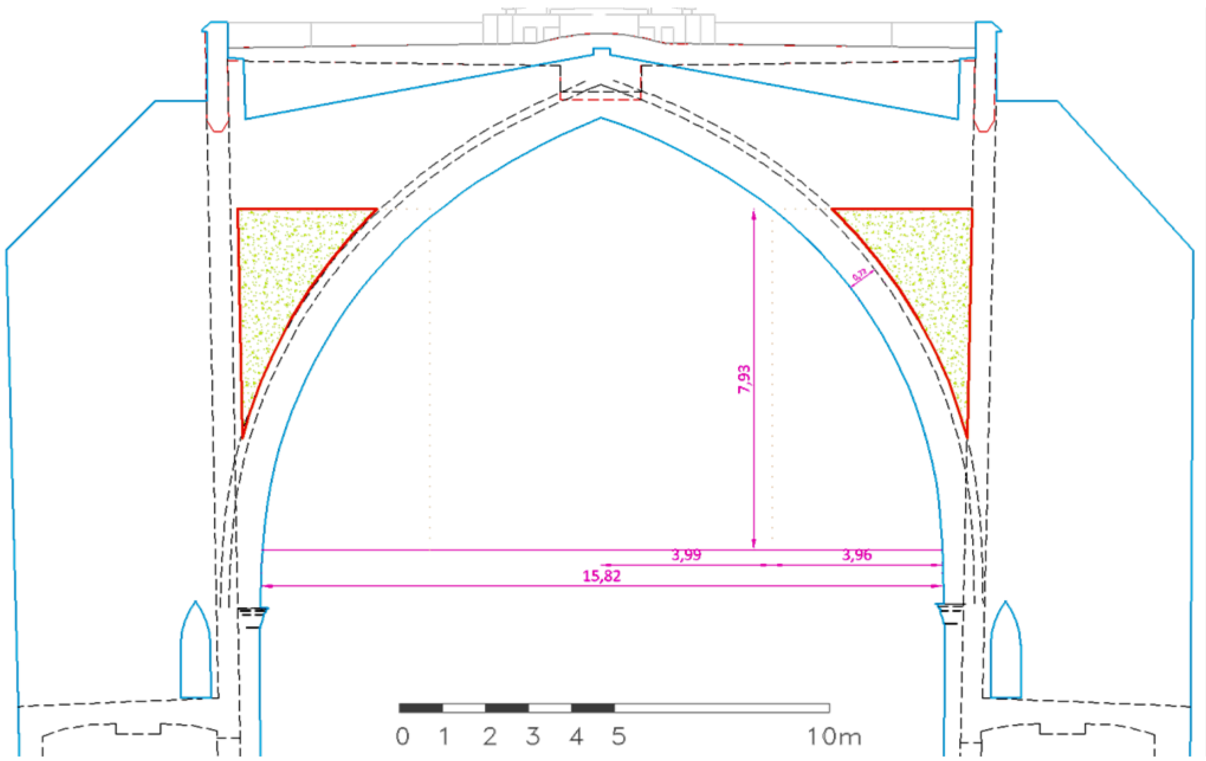



Figure 82 the defined part and height for the sound infill.

#### 5.2.3.6 *The light infill:*

Above the sound infill, another material is used for filling till the roof level. Aggregate, often of pumice which is a light volcanic rock specialized by its high porosity and light weight, is often used as sound infill. In the case of the church of Santa Maria del Pi case (a Catalan structure), some empty pottery combined with mortar was also used for this type of infill. Normally, it has a specific weight which can vary from 4 to 10 KN/$m^3$. I used the value of 4 KN/ $m^3$. The top level of the infill follows certain slopes for the drainage of the rain water and for every cross section; the height varies following the slope.

### 5.2.4 *Model discretization and cross sectioning:*

As previously explained for the different components of the vault, the geometrical shapes and dimensions were derived and defined. The next step was to construct the model. In order to accomplish it, I favored the use of the Google Sketchup software due to its simplicity and practicality. Importing the CAD drawings into Sketchup was simple and accurate. I modeled each component individually with the defined cross section and geometry. Later, I combined all of them together.

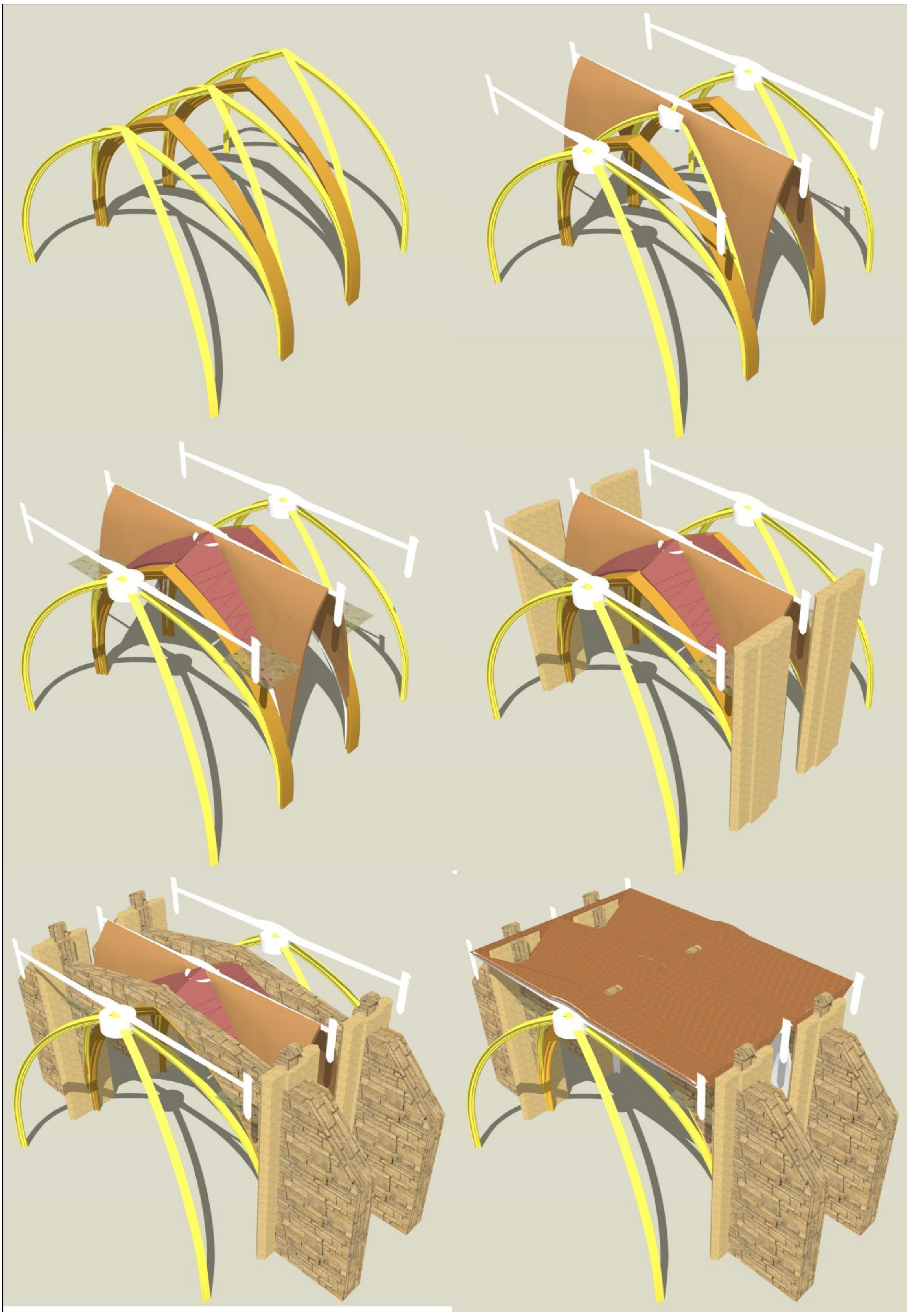


Figure 83 the different steps of construction for the Model.

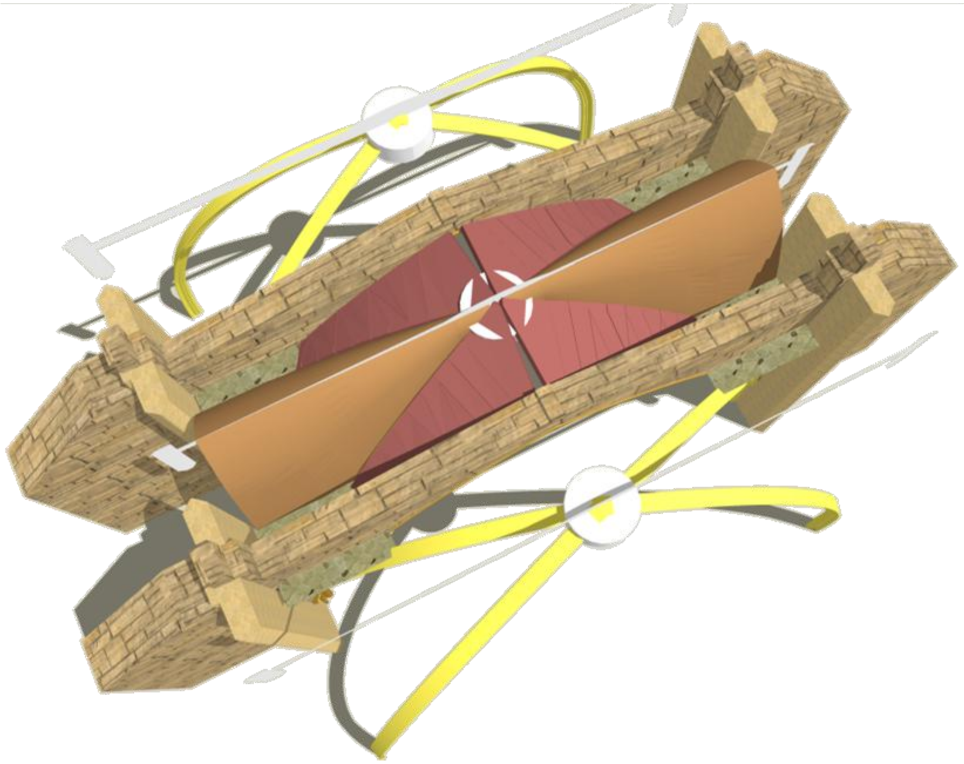


Figure 84 Schematic drawing of the Model

The use of the model had a target of defining the needed cross section planes and using the software to get an accurate geometrical section. It is to be noted that the sections can follow any inclination and can be easily offset, moved or copied. This sliding technique allowed me to get all the cross sections needed for my analytical graphical analysis, as shown in (figure 85-86).

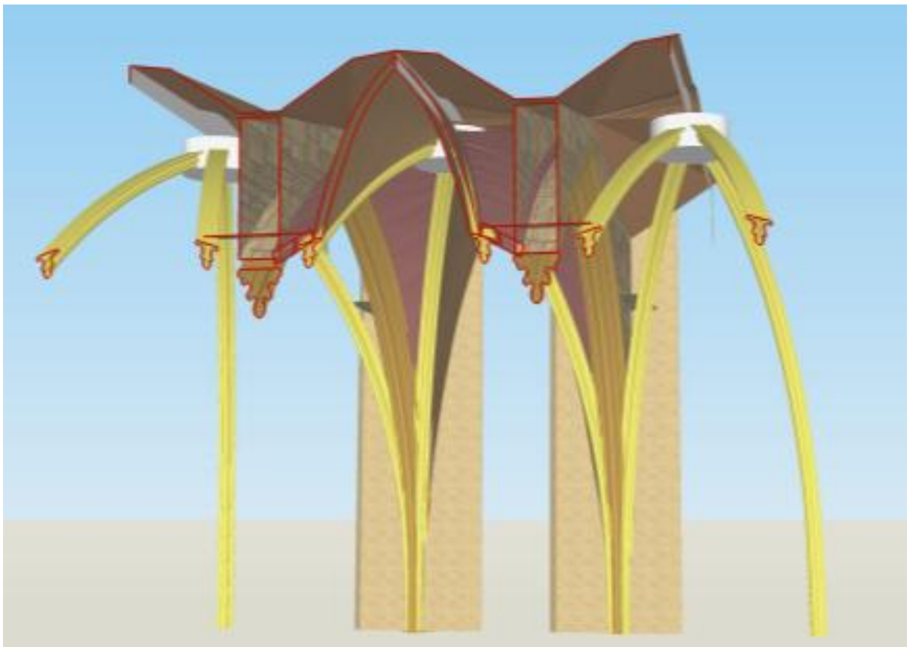


Figure 85 - The use of sketchup for acquiring the cross sections.

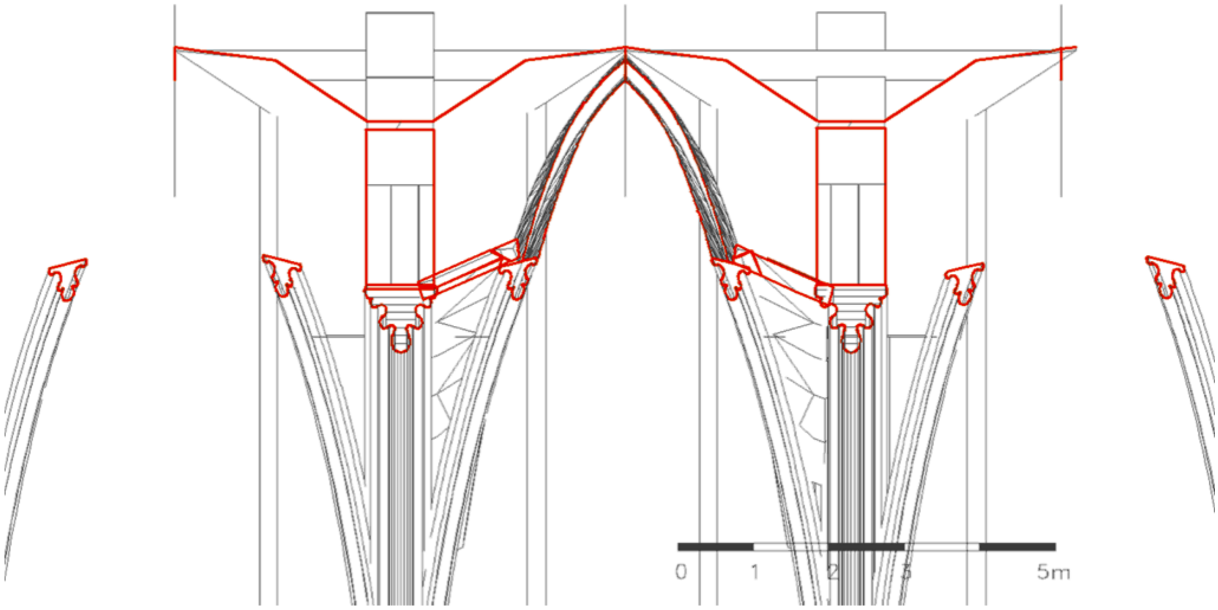



Figure 86 - cross section obtained by Sketchup slicing technique.

### 5.2.5 The design of Excel spreadsheet:

For the analytical analysis of the multiple cross section, I designed an excel template spreadsheet to facilitate the calculations and to draw the line of thrust for checking its instability.

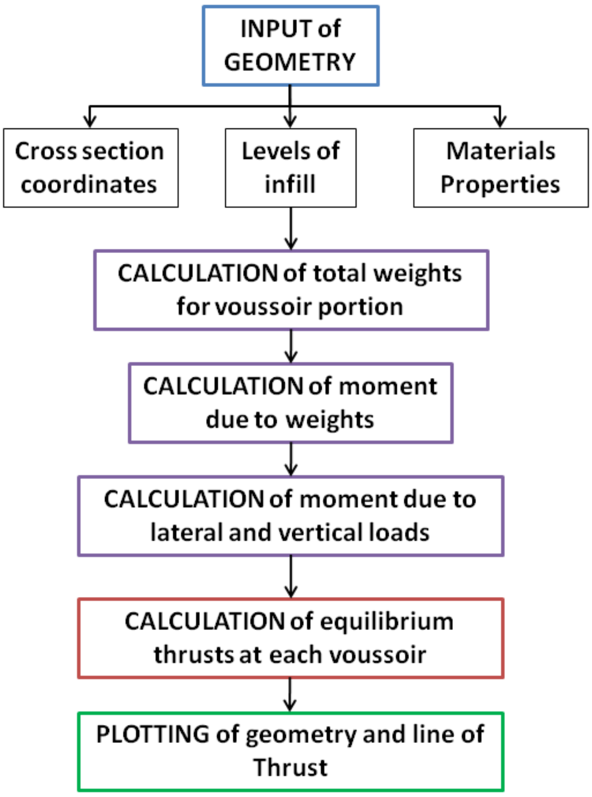



Figure 87 - Process of spreadsheet analysis

The first step was the input of the geometrical coordinates of the cross section into the excel sheet. I favored to input the true geometry of the section instead of using parabolic or circumferential formulas to draw the arch profile in order to have the most accurate representation of the section. By using the Autocad software, I divided the cross section to 41 portions as the excel sheet is to analyse the arch in 41 voussoirs including the above layers of infill. Then by using a special Autocad alias lisp (a computer programming language), I export the arch profile into a list of coordinates (horizontal and vertical values) to the excel sheet and this is done for both intrados and extrados rings, see figure 87.

The input data was: The width of the section (m), The height of the spandrel (m), The infill layers heights (m) and The materials specific weights (KN/m$^3$).

The second step was the calculation of the weights of the different portions of the cross section by calculating the areas for each material within the same vertical portion than they are multiplied by the material specific weight and the thickness of the section to get the total weight of the portion.

Finally, results are plotted including the intrados and extrados rings, the heights of the different infill types combined with the line of thrust for the cross section. For each cross section, a horizontal force and a vertical reaction are determined.

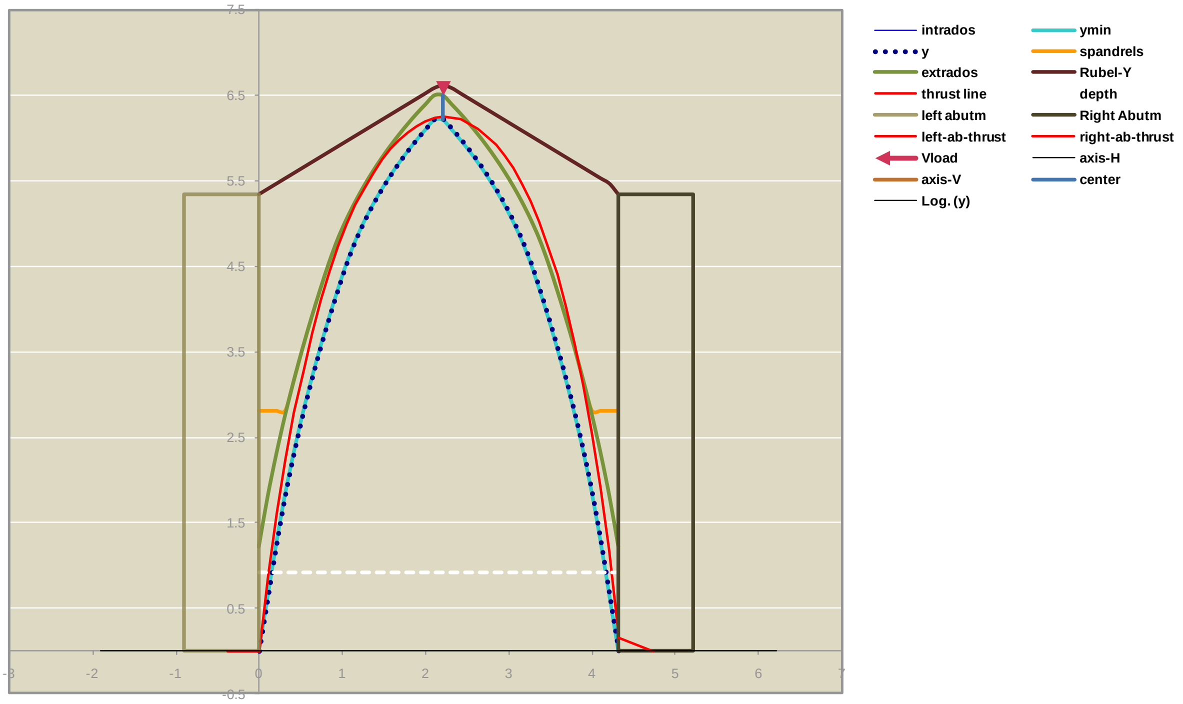



Figure 88 development of the line of thrust using the Excel spreadsheet.

### 5.2.6 *Analysis of the cross sections:*

As I previously mentioned, there are different theories about the behaviour of the vault. This implied the choice of different cross section combinations to analyse the cross vault. In the following section, I will explain each of them.

#### *5.2.6.1 Decomposition of cross vault according to Beranek:*

In this decomposition, as was mentioned by Beranek[5], the diagonal arches (or cross ribs) are critical and receive all the weight of the vault. Thus the cross section combination will be parallel and perpendicular to the main axis of the vault in a way to follow the suggested force trajectories.

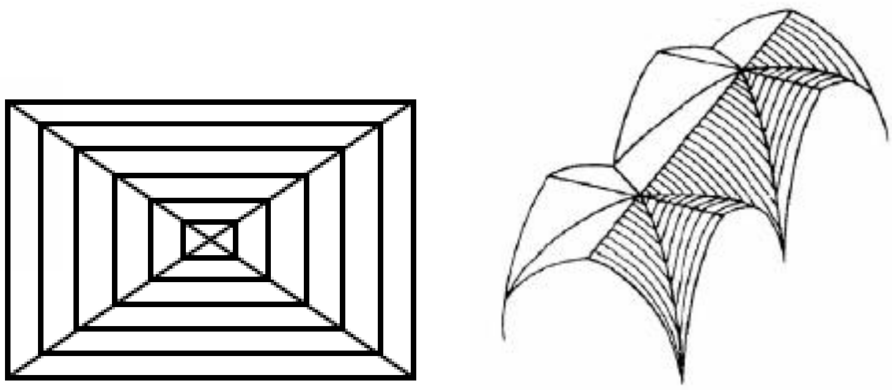


Figure 89 Beranek's force trajectories in a vault subject to dead loading and corresponding division into arches in plan.

Following this approach, I defined a series of longitudinal and transversal sections as shown in figure 89. The transversal sections, mainly slicing the longitudinal vault, have an interval of 0.348 m and are 7 sections named from Tr-sec1 to Tr-sec7. The longitudinal sections, slicing the transversal vault, have an interval of 1m and are also 7 sections named Long-sec1 to Long-sec-7. The difference between the intervals distances is due to the fact that the sections should intersect at the center line of the cross rib. See figure 90.

[5] SAHC 2009 lectures, SA1 module

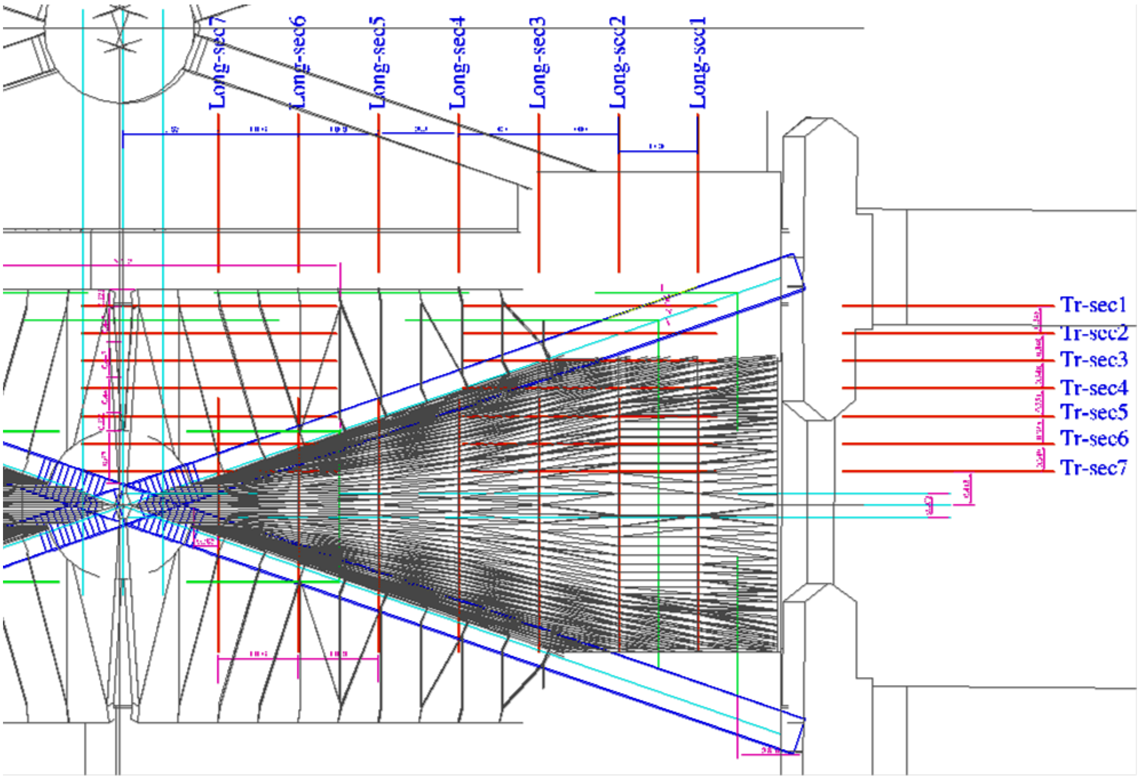



Figure 90 - composition of longitudinal and parallel sections for SM del Pi.

Each arch of the mentioned cross section is analyzed in an individual spreadsheet and the line of thrust is drawn for the maximum horizontal force achieving equilibrium. It is to be noted that I applied a vertical load on the top of extrados ring and at the middle of the arch span. This load represents an added material having the same specific weight of the stone and that is applied to provide a pressure required for the stability of the arch. This load is calculated by defining a horizontal strip at the top of the vault and a variable height to each section (depending on the height of the infill at the middle). The volume of this part is calculated and multiplied by the specific weight to get the value of the load. In the case of the lateral vault, I assumed the width of this strip is equal to 0.3m and for the longitudinal vault, I assumed the width equal to 1m. The difference of width is clarified by the fact that with the change of the arch span, a higher load for stabilization is required and the width for the portion of stone should be relative to the span of the arch. Figures 48 to 51 show the carried on analysis for simultaneously Long-sec1 and Trans-sec1, each section is represented in CAD and in Excel with its respective line of thrust.

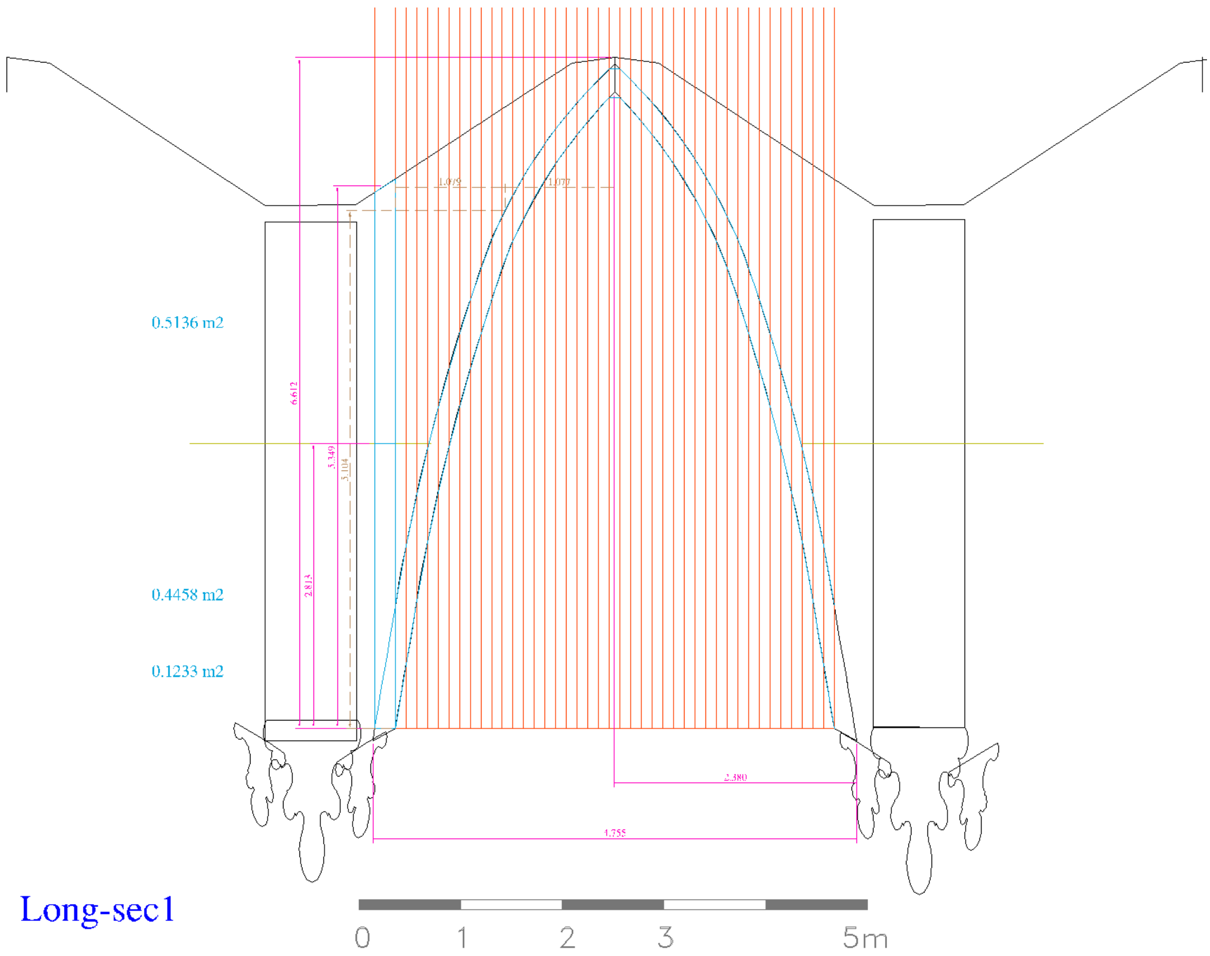



Figure 91 - Long-sec1 CAD geometry and dimensions.

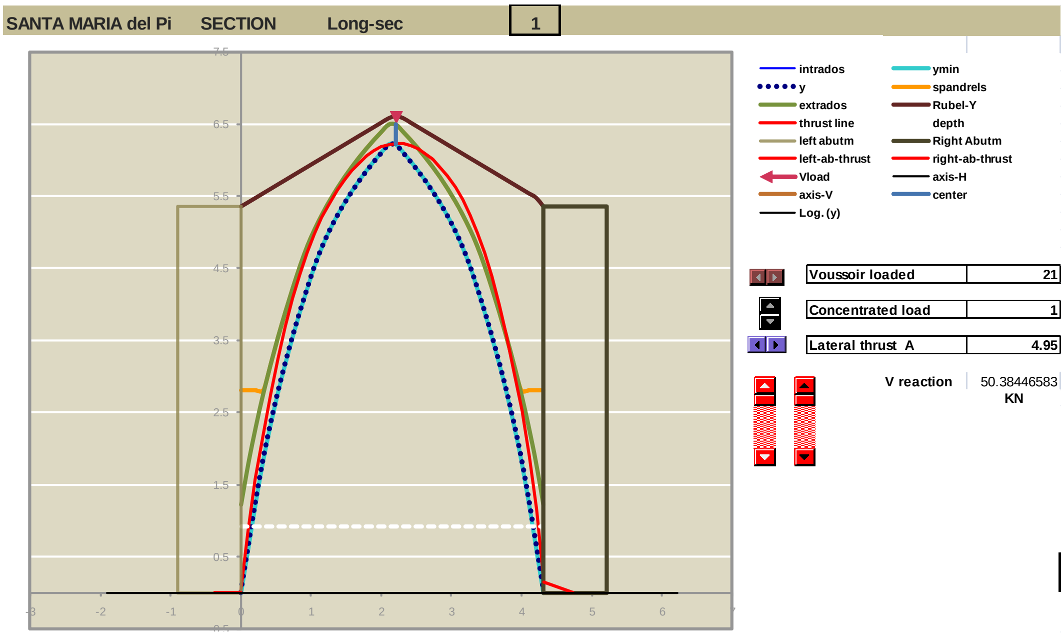



Figure 92 - Long-sec1 Excel Line of thrust.

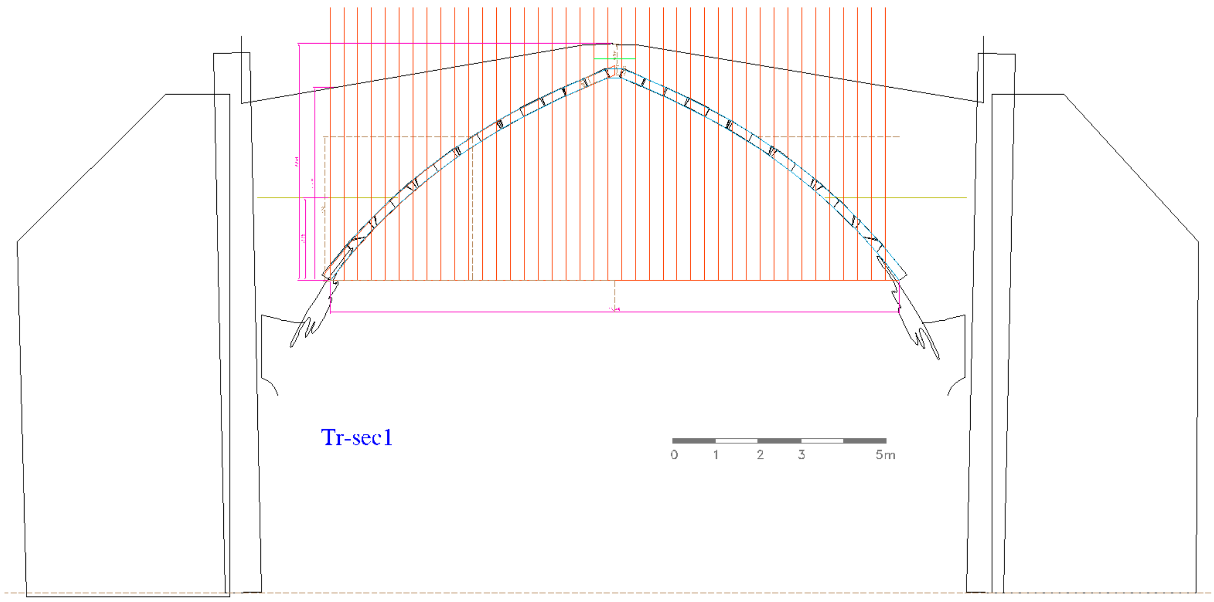



Figure 93 Trans-sec1 CAD geometry and dimensions.

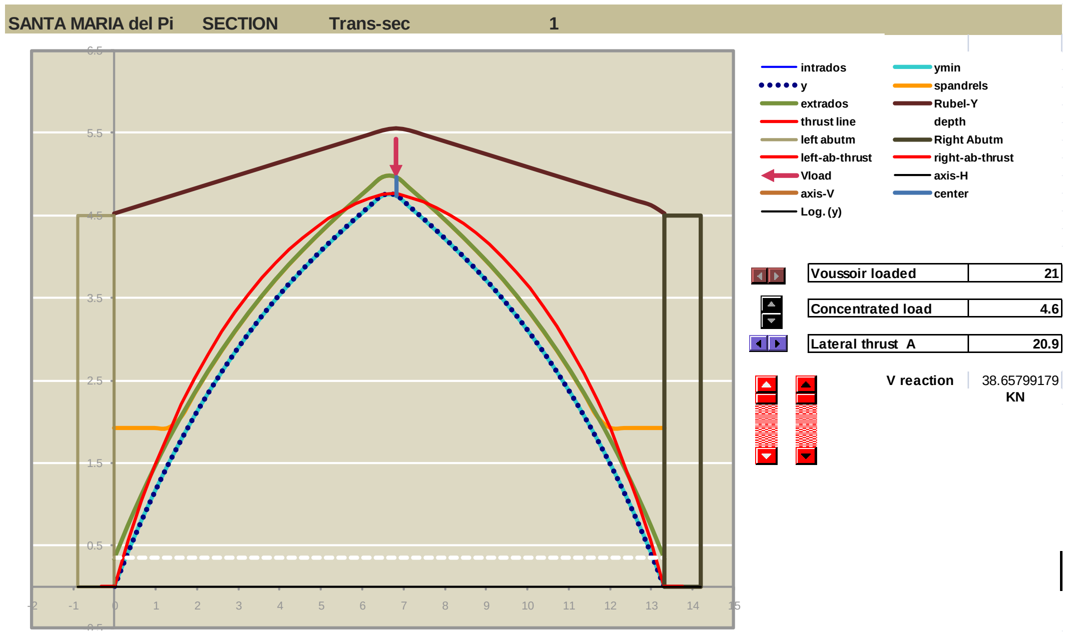



Figure 94 Trans-sec1 Excel Line of thrust.

As can be seen in figures 92 and 94, a line of thrust can exist within the voussoirs of the transversal vault and for its different analyzed cross sections, but for the longitudinal vault, and all its analyzed sections, the line of thrust starts within the arch then is always getting above the extrados ring and at the middle of the span goes back again within the arch.

After analyzing the cross sections for both vaults, I determined the horizontal force and the vertical reaction for each cross section. And as mentioned before, each longitudinal and transversal section are intersected in one point along the axis of the cross rib. I add the values of the vertical loads for the linked sections. For example, the vertical value for Long-sec2 is added to the vertical value for trans-sec2. And this is done for all the sections.

| | | | | | |
|---|---|---|---|---|---|
| | vertical | loads | | | |
| | | | | | |
| 9.27 | | | | | |
| | 13.76 | | | Longitudinal Vault | |
| 11.49 | | 23.27 | | | |
| | 17.5 | | 26.69 | | |
| | | 25.1 | | 38.65 | |
| | | | 32.85 | | |
| transverasl Vault | | | | 50.38 | |
| | | | | | |
| | | | | | |
| **20.76** | **31.26** | **48.37** | **59.54** | **89.03** | |

Figure 95 - sum of vertical loads.

As for the horizontal forces, I followed a different procedure. For each section, I decomposed the horizontal force in two components; one in the direction of the cross rib axis and the other in the perpendicular direction to the cross rib axis. This was done using the angle between the cross rib and the sliced section of the longitudinal vault and which is equal to 20° for the case of the intersected vault of Santa Maria del Pi.

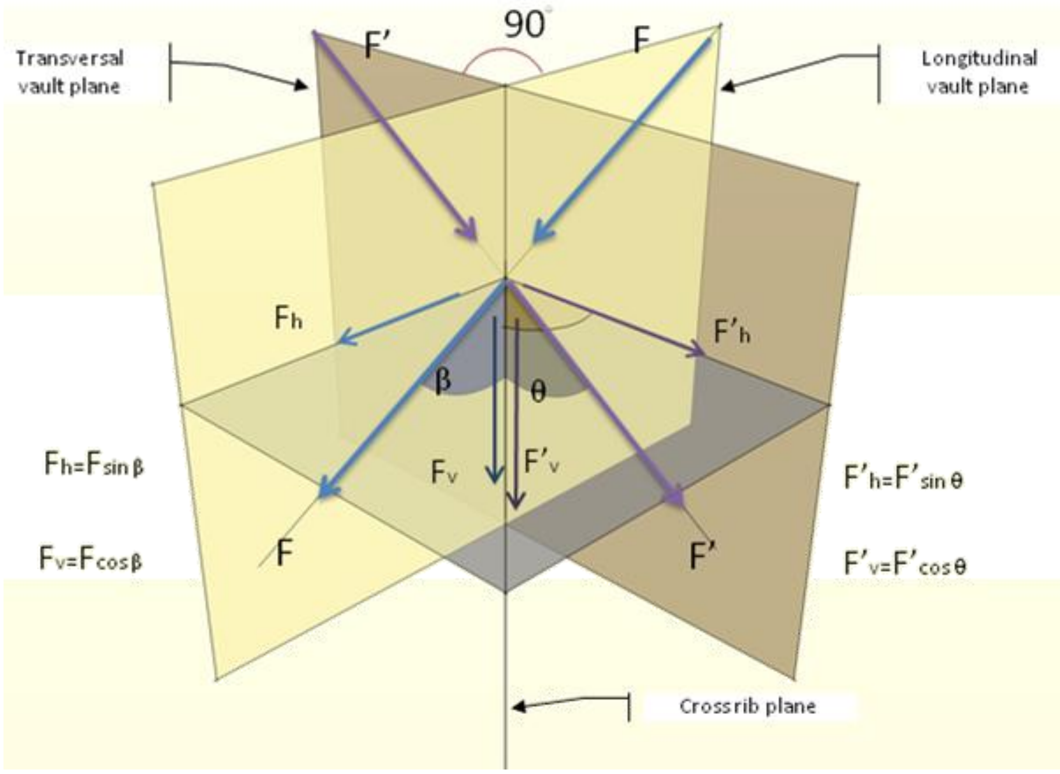



Figure 96 - Decomposition of forces for two perpendicular planes.

Supposing that F is a resultant force in the plane of the longitudinal vault, and F' is another resultant in the plane of the transversal vault where $\beta$ *and* $\theta$ are respectively the angles of these resultants to the vertical axis. See figure 96. Fv is the vertical component of F and equal to Fcos$\beta$ and similarly F'v is the vertical component of F' and equal F'cos$\theta$. It is to be noted that the horizontal and vertical components of the resultant force for each vault section was directly calculated from the spreadsheet and I didn't need to calculate them separately following the previous process. The total vertical force applied in the cross rib plane will be Frv=Fv+F'v. For the other 2 components Fh and F'h, they will be acting in a horizontal plane to the cross rib plane. Again, each of them should be decomposed in two different components; one in the plane of the cross rib and another perpendicular to the cross rib plane. This is done for Fh and F'h as follows;

F'ha which is perpendicular to the cross rib plane is equal F'hsinα and F'hb in the plane of the cross rib is equal to F'hcosα, where α is the angle between the cross rib plane and the plane of the transversal vault resultant (same plane of its horizontal component F'h). Same procedure is done for the other horizontal component Fh which will have Fha=Fhcos α and Fhb=Fhsin α. This is shown in figure 97.

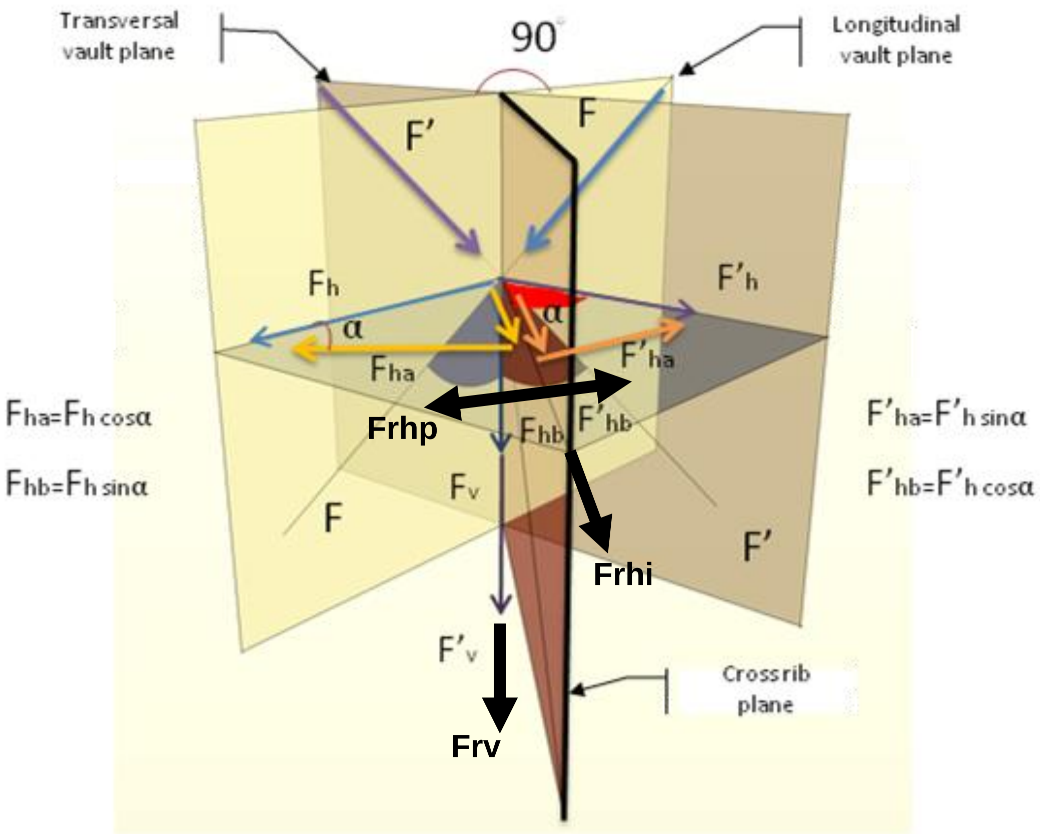



Figure 97 - decomposition of forces relative to the cross rib plane.

Thus, I get two horizontal forces. The first force Frhi=Fhb+F'hb and this force is acting horizontally in the same plane of the cross rib. The second force is Frhp=Fha-F'ha and is acting perpendicular to the plane of the cross rib. Theoretically, this force Frhp should be equal to zero to achieve equilibrium, however, it is not really true as this out of plane force can exist and be counter acted by an opposite force in the global structural system.

| | | | | | | |
|---|---|---|---|---|---|---|
| | horizontal loads | | | angle | 20 | |
| | | | | | | |
| comp1 | 10.7 | | | | | |
| | | 13.2 | | | Longitudinal Vault | |
| comp2 | 2.3 | | 15.8 | | | |
| | | 3.15 | | 17.6 | | |
| | | | 4.3 | | 20.9 | |
| | | | | 4.4 | | |
| | transverasl Vault | | | | 4.9 | |
| | | | | | | |
| linear comp 1 | 4.366478 | 5.386683 | 6.447697 | 7.182244 | 8.528915 | |
| linear comp 2 | 2.099774 | 2.875778 | 3.925665 | 4.016959 | 4.519079 | |
| SUM | 6.466252 | 8.262461 | 10.37336 | 11.1992 | 13.04799 | |
| | | | | | | |
| perp comp 1 | 9.768514 | 12.05088 | 14.42453 | 16.06784 | 19.08056 | |
| perp comp 2 | 0.938589 | 1.285458 | 1.754753 | 1.795561 | 2.020006 | |
| ratio | 10.40766 | 9.37477 | 8.220266 | 8.948644 | 9.445791 | |

Figure 98 - components of lateral forces and their sum.

The previous procedures are carried out to determine the forces applied by both vaults on the diagonal cross rib. Again, I performed the same cross section procedures to analyse the cross rib and taken into consideration all the vertical loads and horizontal forces caused by the longitudinal and transversal vaults.

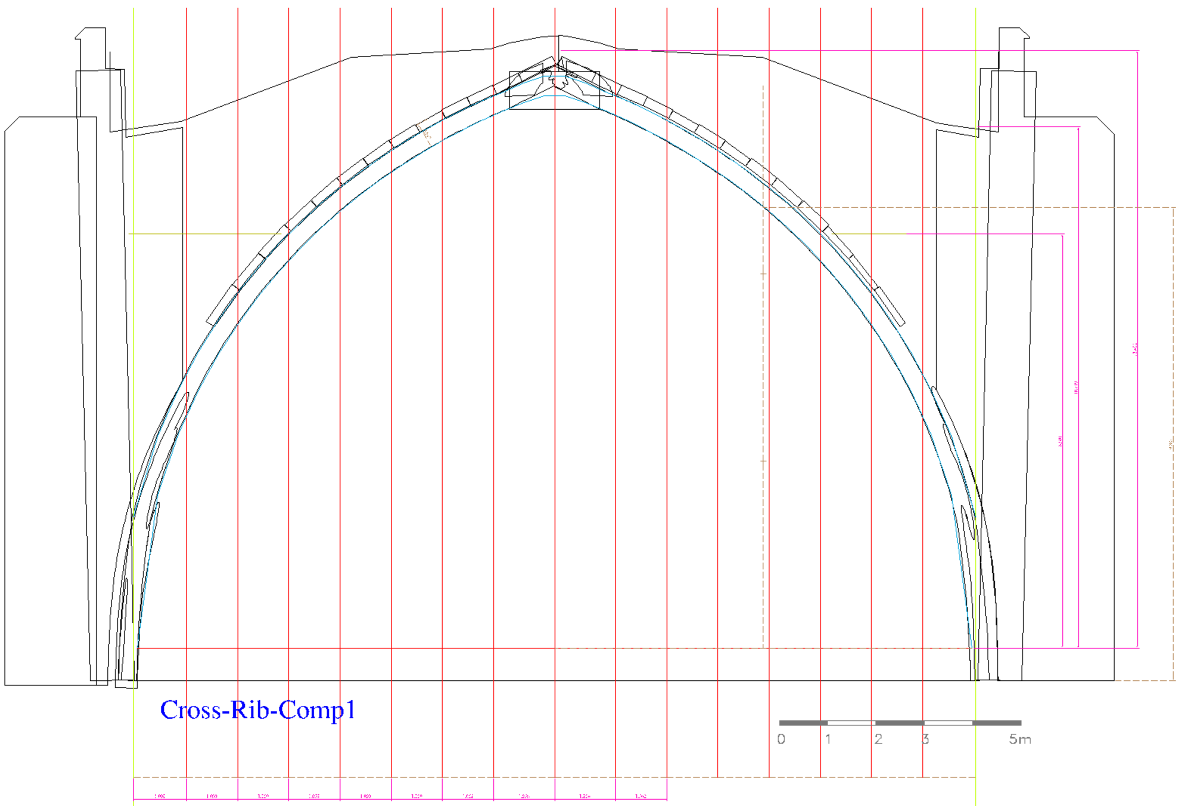


Figure 99 - cross rib CAD geometry and dimensions.

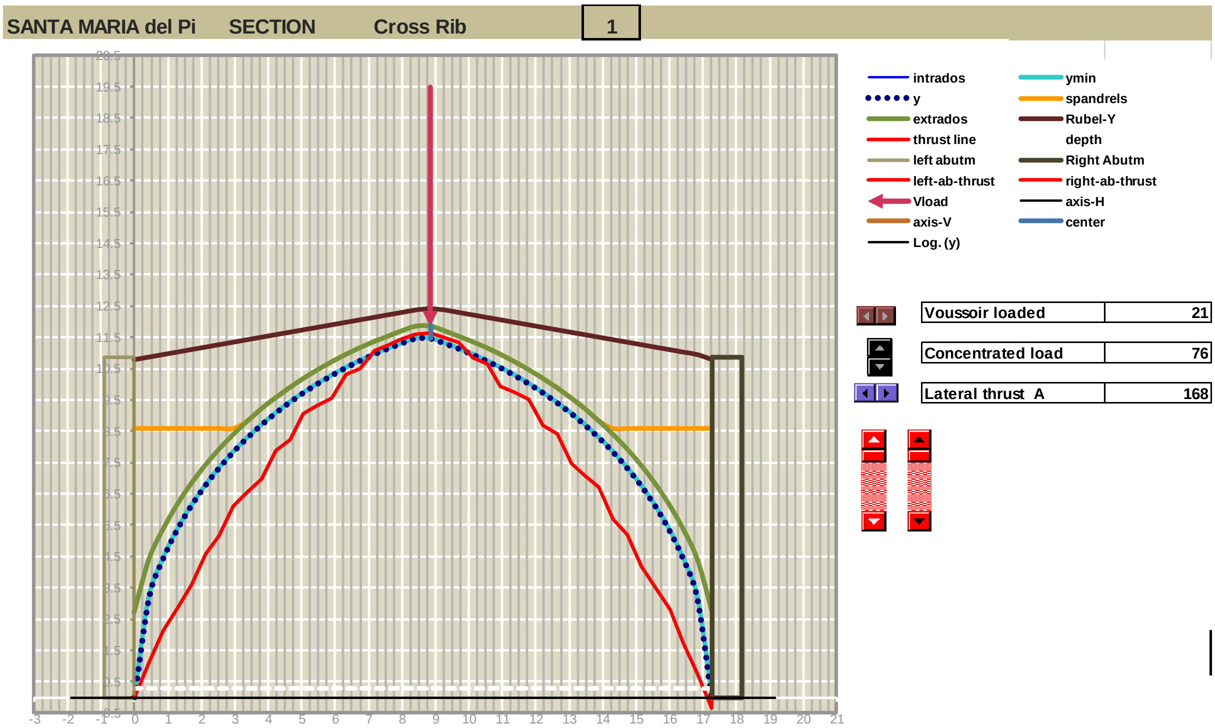



Figure 100 - cross rib section with the resultant line of thrust

As can be seen in the figure 100, the graphical analysis by using the excel spreadsheet shows that the line of thrust is far below the intrados ring and it is only in the middle of the span that the line is located between the intrados and extrados rings. There is no line of thrust that can be passing through the arch voussoirs of the cross rib.

### *5.2.6.2 Decomposition of cross vault according to Mark:*

In this decomposition, as was mentioned by Mark, the membrane is the critical member; while the diagonal arches (or cross ribs) do not work. Thus the cross section combination will be diagonal to the main axis of the vault in a way to follow almost the suggested force trajectories. It is to be noted that this approach was favored to be more realistic especially for late Gothic vaults with weak diagonal arches (cross ribs).

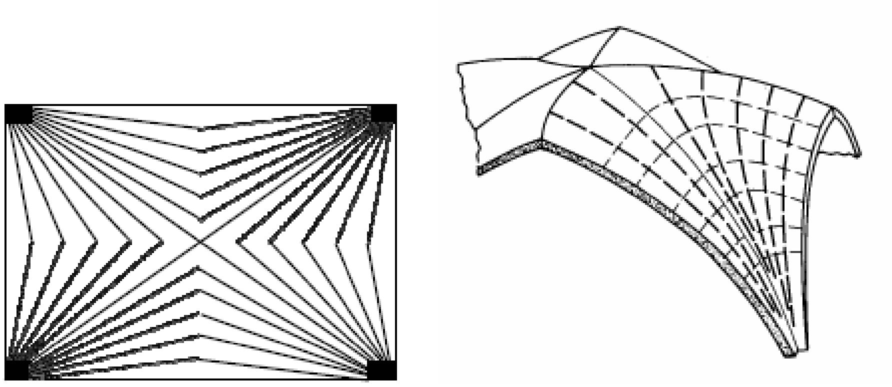


Figure 101 - Mark's Force trajectories in a Gothic vault subject to dead loading and corresponding division into arches in plan

The theory of Sabouret cracks and carried finite element model analysis and calculations suggest that the cross vaults might behave in an additional composition. Especially for a rectangular plan cross vault, that has one side longer than the other. The longer part acts as one dimensional membrane and can develop diagonal arches. The other shorter part would develop parallel arches supported on the first or on the cross rib.

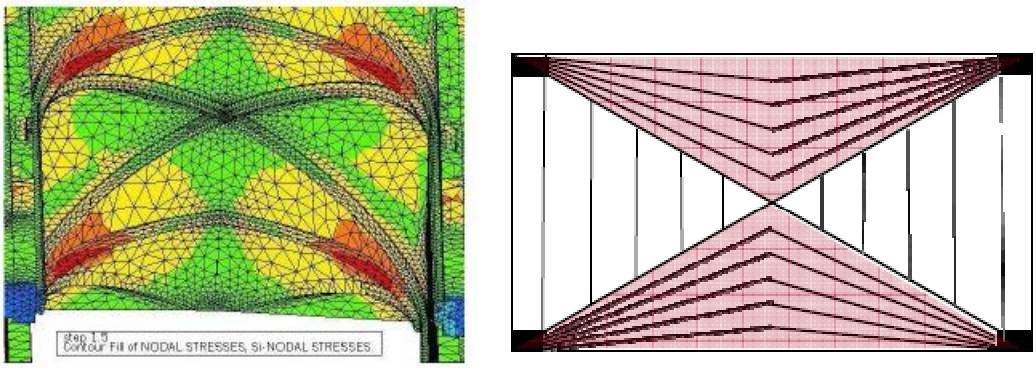



Figure 102 - decomposition of parallel and diagonal sections.

Following this second approach, I defined a series of longitudinal and diagonal sections as shown in figure 103. The longitudinal sections, mainly slicing the transversal vault, have an interval of 1 m and are 7 sections named from Long-sec1 to Long-sec-7. The diagonal sections, slicing the longitudinal vault, are also 7 sections named D-sec1 to D-sec-7. However, they have an interval of 0.296 m at the middle of the vault and they coincide in one point at the side of the longitudinal wall. This previous point is exactly located at the intersection between the transversal nave arch and the cross rib. See figure 60.

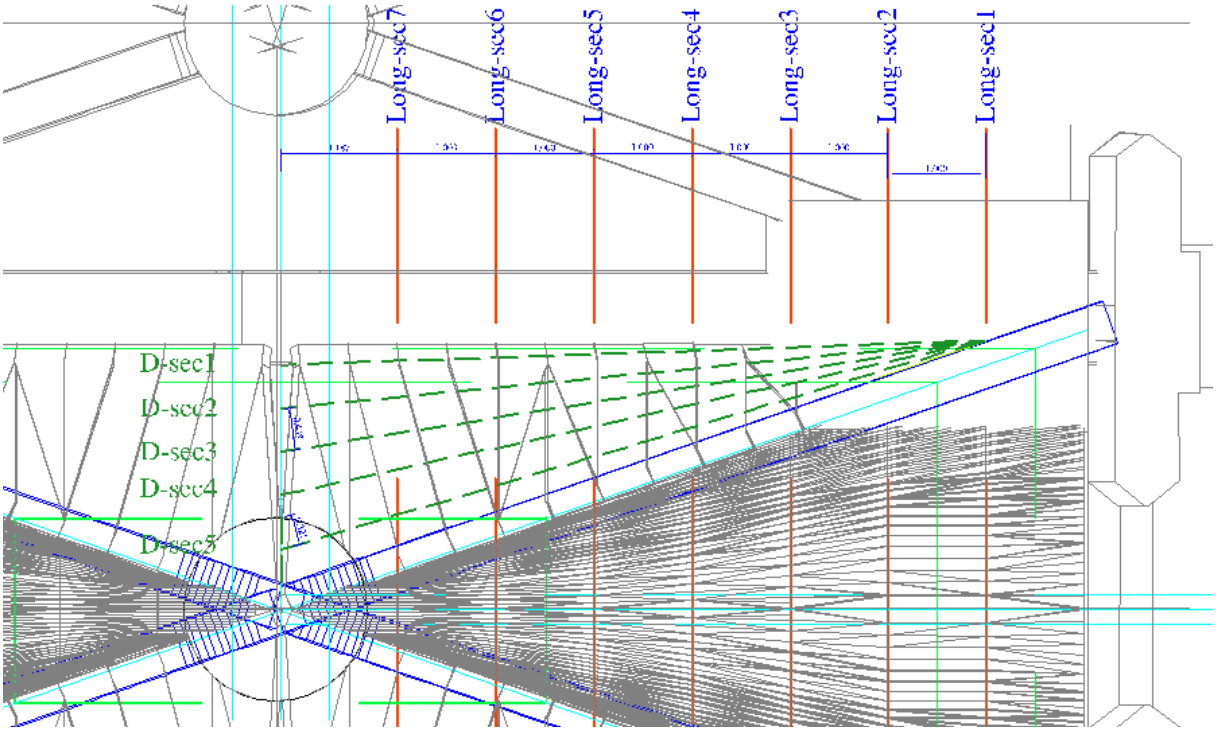



Figure 103 - decomposition of diagonal and longitudinal sections for SM del Pi.

This implies that the analysis of the cross section has not a constant thickness but the width should vary depending on the horizontal coordinate of the section. I took in consideration the previous change by applying an equation defining the width as follows; w =width*2*C52/span for each voussoir where

*Width = the maximum width of the cross section located at the middle of the span.*

*C52 = the horizontal coordinate value of the voussoir in the first half.*

*Span = the total span of the cross section.*

The previous equation is for the first half of the span and as for the second half, I use another equation;

W=width*2*(span-C75)/span where

*Width = the maximum width of the cross section located at the middle of the span.*

*C75= the horizontal coordinate value of the voussoir in the second half.*

*Span = the total span of the cross section.*

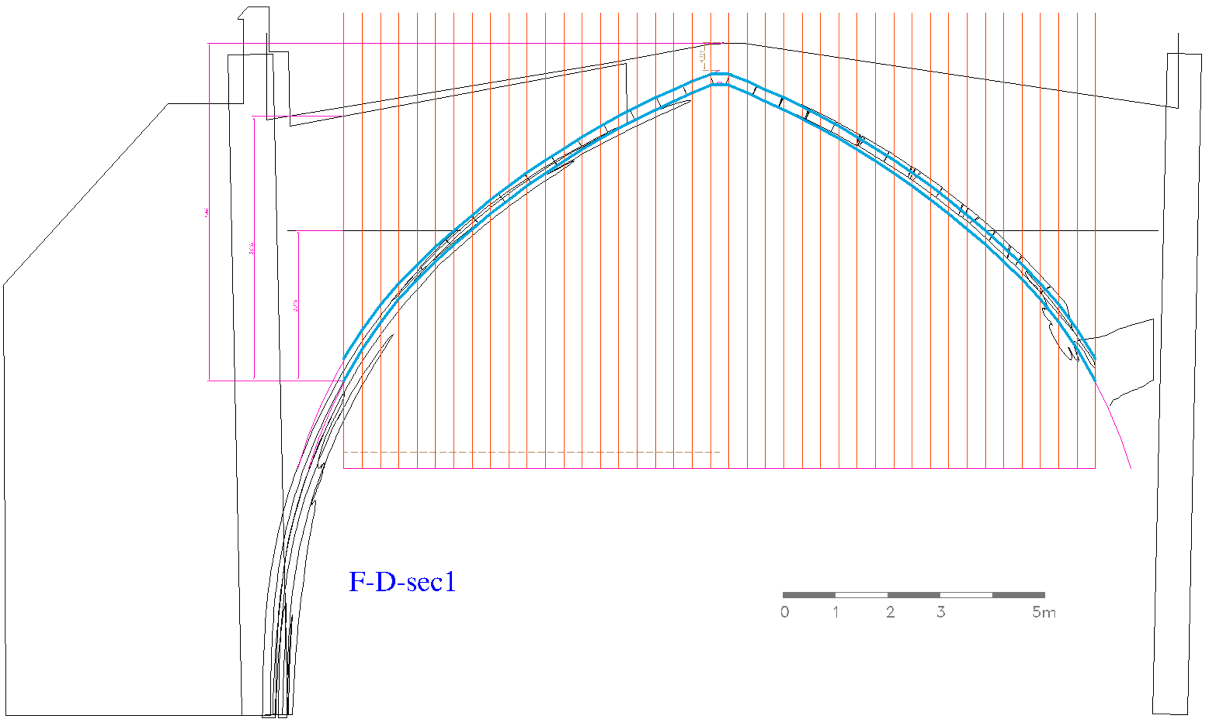



Figure 104 - D-sec1 CAD geometry and dimensions.

It is to be clarified that the diagonal cross section, developed by the Sketchup slicing technique and imported to the CAD software, was not symmetrical as the slicing plane passed only by one half of the longitudinal vault. I had to correct the section manually by mirroring the half span section.

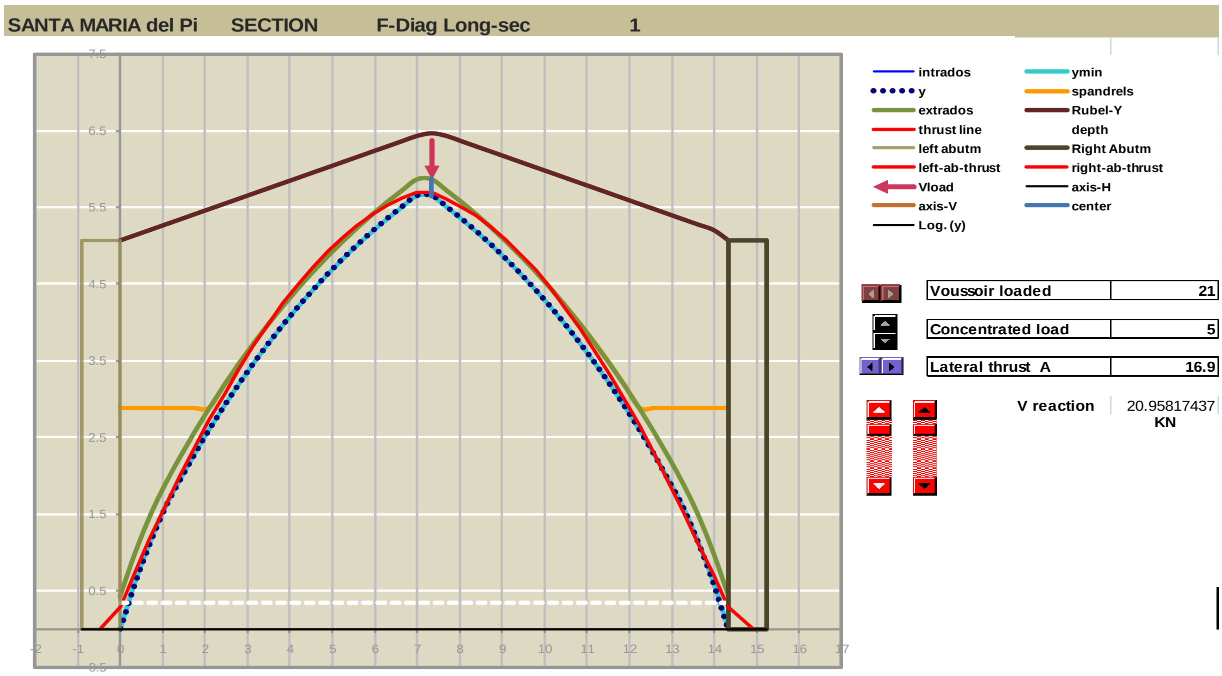



Figure 105 - D-sec1 Excel Line of thrust.

From the analysis shown in figure 105, in all of the analyzed sections, the line of thrust was almost within the region of the arch in the first quarter of the span and then goes slightly to the exterior light infill and then back within the arch till the middle of the span.

As for the longitudinal sections slicing the transversal vault, I reused the previously defined sections and analyzed in the first decomposition were used. And for the analysis of the cross rib section, I only used the loads imposed by the transversal vault. The diagonal sections of the longitudinal vault transmitted their thrusts at the connection of the transversal nave arch and the cross rib to the adjacent abutment.

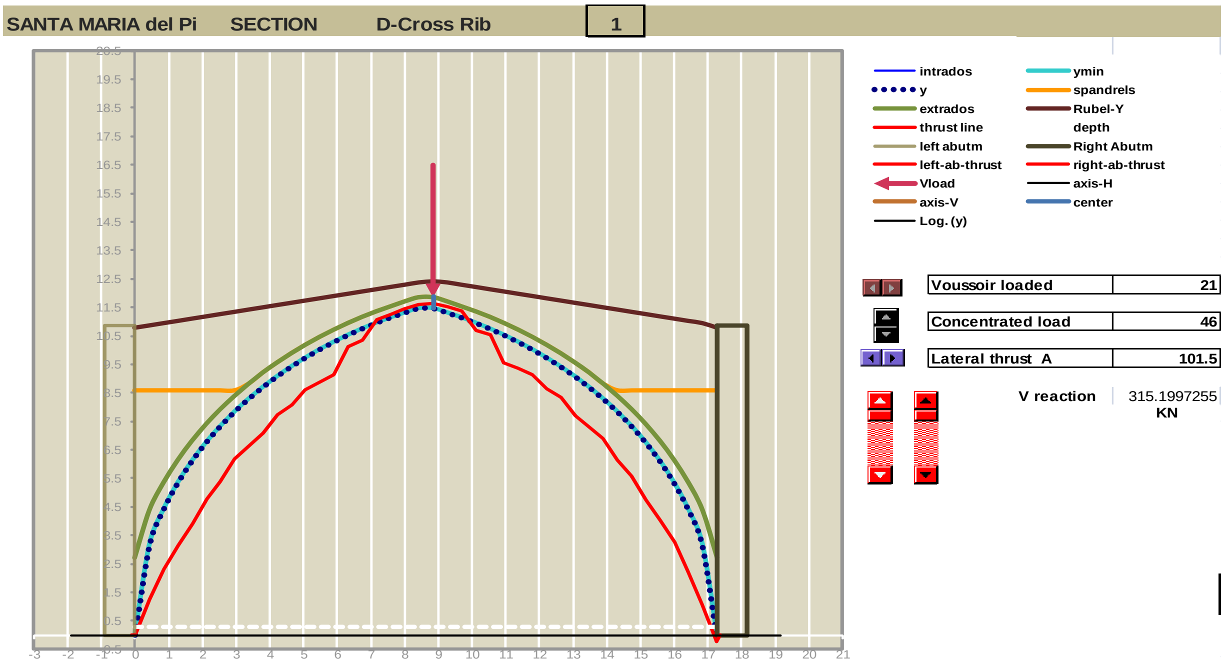



Figure 106 - the line of thrust resultant of cross rib section associated with the diagonal decomposition.

As shown in figure 106, for most of the span of the cross rib, the line of thrust was totally out of the intrados and extrados rings of the arch voussoirs. Only for a small part, the line of thrust is within the designated region. The combination of forces acting on the cross rib, vertical and lateral, cannot develop a line of thrust that follows the shape of the analyzed arch.

Moreover, I analyzed the cross rib section with neglecting all the vertical and lateral forces imposed by the transversal and longitudinal vaults. Only the own weight of the cross rib and the weights of the infill layers were considered. In this analysis, the line of thrust was also out of the stone voussoirs arch region for most of the span and only inside the region in a very small distance at the middle of the span. This is clearly seen in figure 107.

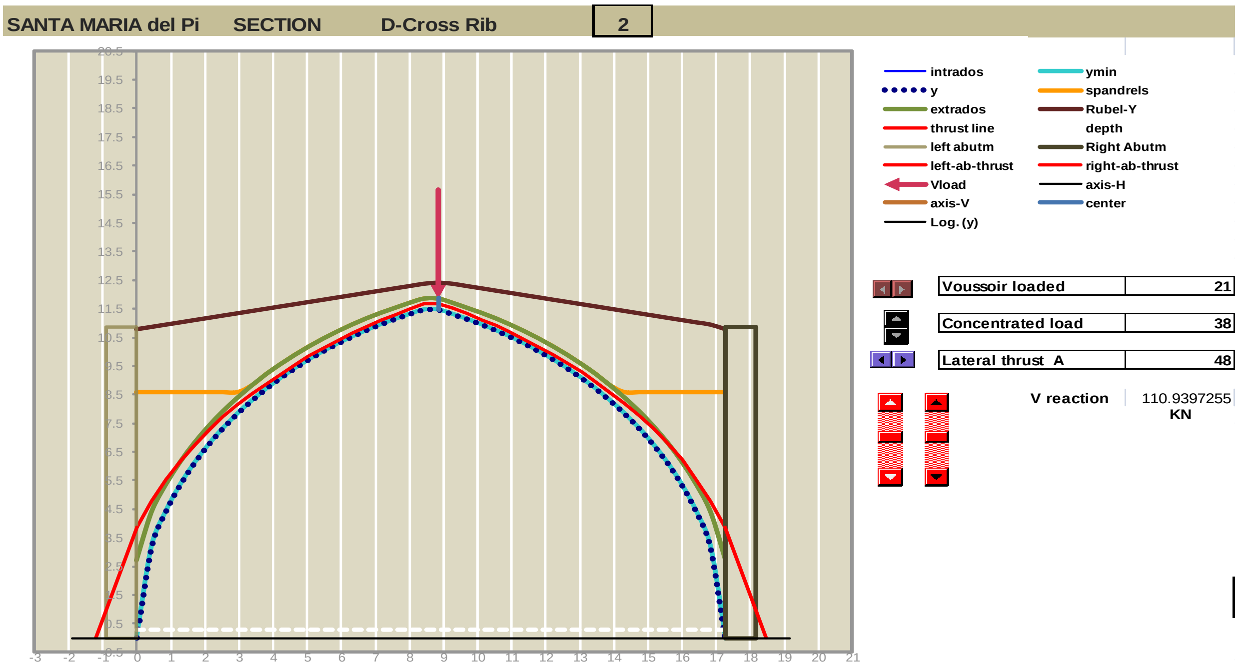



Figure 107 - Analysis of the cross rib considering only its own and infill weights

### *5.2.6.3 Decomposition of the cross vault by the researcher:*

I explained in my analysis the main two different theories of decomposing the cross vault and their corresponding presumed force trajectories. However, as previously explained, in both decompositions, I couldn't find a convincing line of thrust that exists within the intrados and the extrados rings of the arches. Thus, I decided to follow a new and different decomposition process for the longitudinal vault and to analyse the related sliced cross sections.

In this decomposition, I choose a sequence of diagonal arches, always keeping a symmetrical order, but these sections are parallel to the cross rib and to each other. They have a constant interval at the side of the transversal nave arch and having different constant interval at the longitudinal axis of the church. The constant interval for the parallel sections is equal to 0.425 m as shown in figure 66. The proposed sections, slicing the longitudinal vault, are 5 sections named IM-sec1 to IM-sec-5 and they are shown in figure 108.

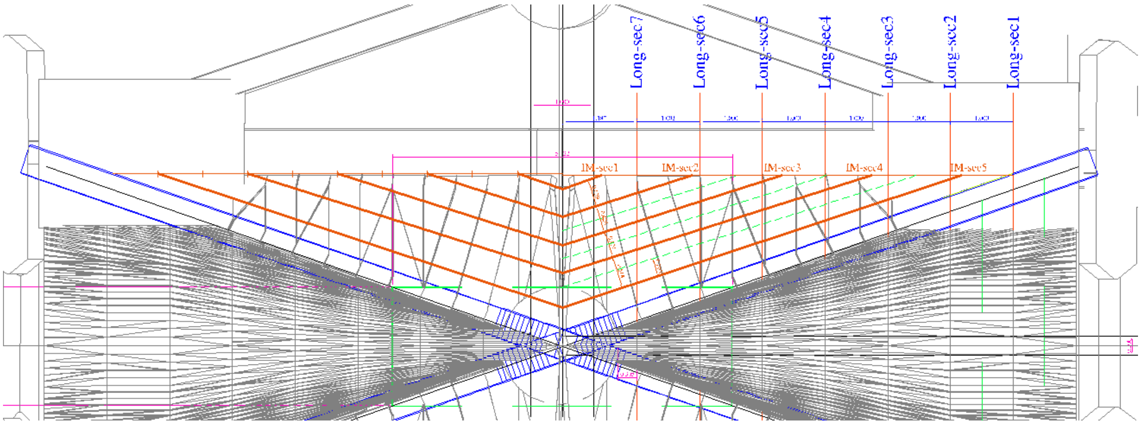



Figure 108 - the proposed decomposition of the longitudinal vault.

As the slicing plane, developed by the Sketchup slicing technique and imported to the CAD software, passed only by one half of the longitudinal, I had to correct manually the proposed cross sections of the vault by mirroring the half span section. See figure 109.

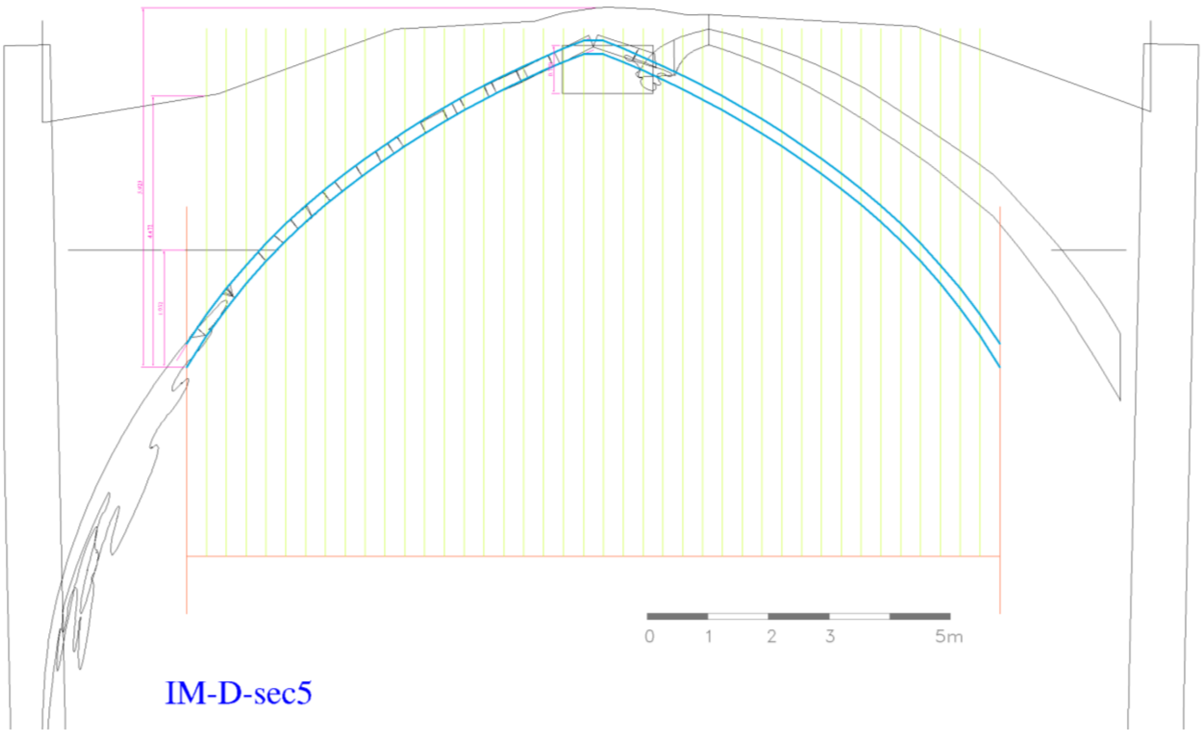



Figure 109 - cad drawing for the proposed section IM-D-sec5.

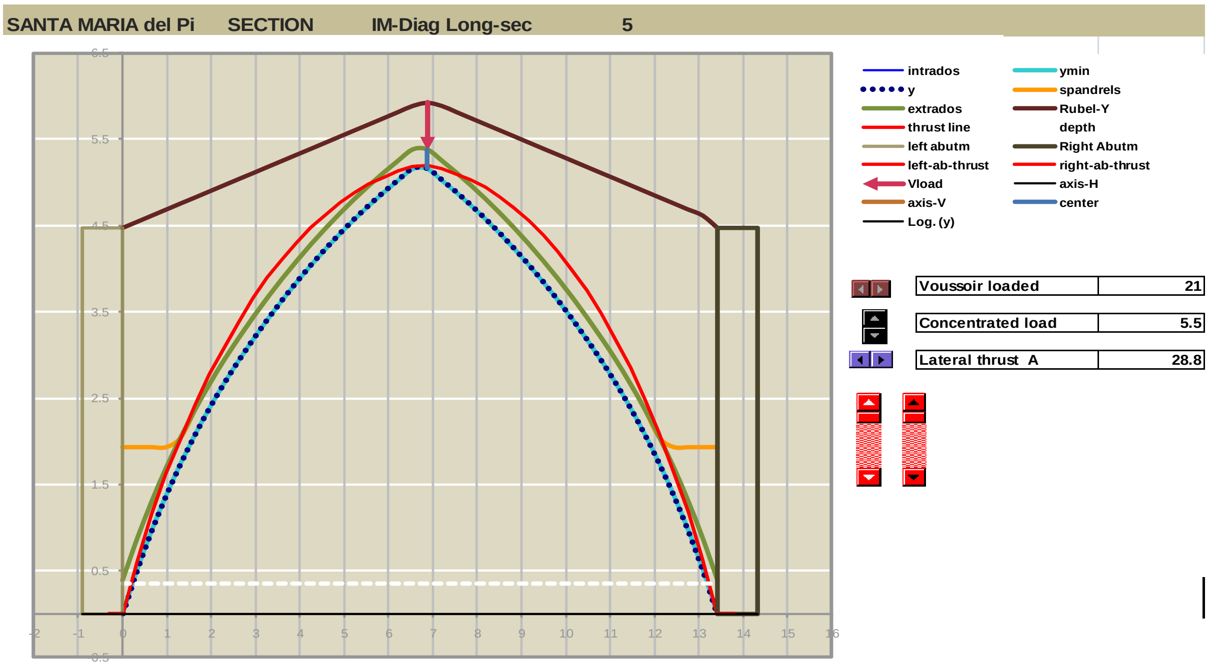



Figure 110 - Excel analysis of IM-D-sec5

From the graphical analysis using the spreadsheet, I didn't find a line of thrust passing only within the region of the arch stone voussoirs and it went out through the light infill.

### *5.2.7 Observations and Results:*

For the observations regarding the conducted analysis, these are the main observations;

1- In Beranek's decomposition, it was possible to find some lines of thrust to pass through the stone voussoirs of the transversal vault sections, however they don't indicate any stability and they represent the development of 5 points hinge consequently the presence of a failure mechanism for this vault.

2- Also, in the same decomposition, for the analysis of the selected cross sections of the longitudinal vault, it was not possible to find a line of thrust passing by the stone voussoirs of the vault.

3- As for the cross rib, the analysis proves that it is impossible to find a line of thrust within the voussoirs region.

4- In the Mark's decomposition, the analysis of the diagonal cross sections for the longitudinal vault failed in realizing a line of thrust passing by the stone voussoirs of the vault.

5- In the third decomposition proposed by the researcher, again no line of thrust only passing through the region of the stone voussoirs of the arch was found.

6- In all analyzed compositions, it was not possible to find a line of thrust passing only by the cross section of the arch stone voussoirs. The obtained lines always went outside to the outer infill; usually to the light infill and a few times to the sound infill.

7- The only possible condition to achieve the required line of thrust is by increasing the concentrated load on the middle top of the ring of the arch. However, this requires unrealistic high values exceeding 10 KN and cannot be clarified.

Results:

- The decomposition of gothic cross vaults following Beranek theory cannot be true as the cross rib cannot support the applied loads of the transversal and longitudinal vaults.
- The vaults membranes composed of thin stones and having a pointed arch shape cannot support individually the combination of loads consisting of their own weights and the other loads applied to them. The existence of a convincing line of thrust through the voussoir arches was not possible in all analyzed sections. However, the structure is standing with no collapse which affirms a state of equilibrium. This can only be explained by the fact that the line of thrust goes outside of the arch to the exterior infill of the spandrel. This infill material has a reasonable compressive strength that can support the stresses developed by the arches of the vaults. Figure 111 shows the analysis of the cross rib, where only its own weight and the weights of the infill were applied, and with a line of thrust located in the region of the voussoirs and crossing through the sound infill (spandrel).

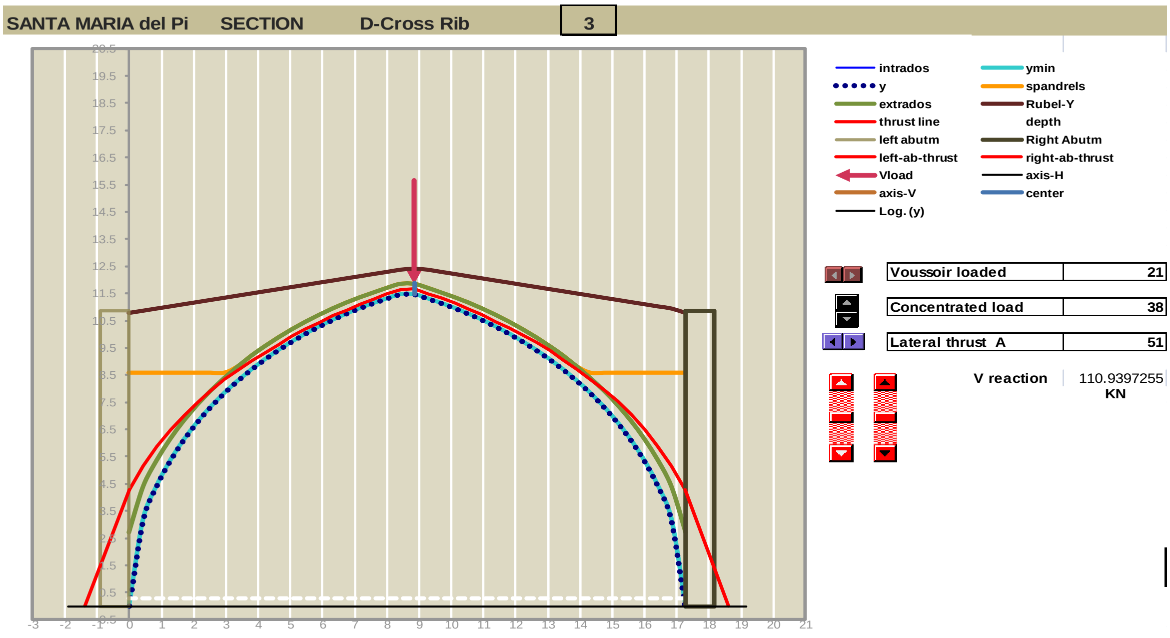



Figure 111 - analysis of the cross rib showing a line of thrust going through the sound infill

- The lateral thrusts passing by the spandrel infill should be transferred to the foundations and later to the soil (ground). By returning to the geometry and the plan of the church, we can see that the transversal nave arch and the cross ribs are concentric in almost one point at the side wall. And exactly at this point is located the buttress with quiet a bulky dimensions. It has a width of 1.52 m and length of 4.8 m not including the thickness of the side wall. This massive and stiff mass was intended to counter act the accumulation of the lateral thrust of the structure and thus providing the necessary needed equilibrium. This is shown in figure 68. Also, the fact that the buttress height (26.75m) reached the top level of the transversal arch crown can mean that more weight was needed and also more vertical height to withstand the lateral thrust. Figure 112 shows a cross section of the side wall and the abutment.

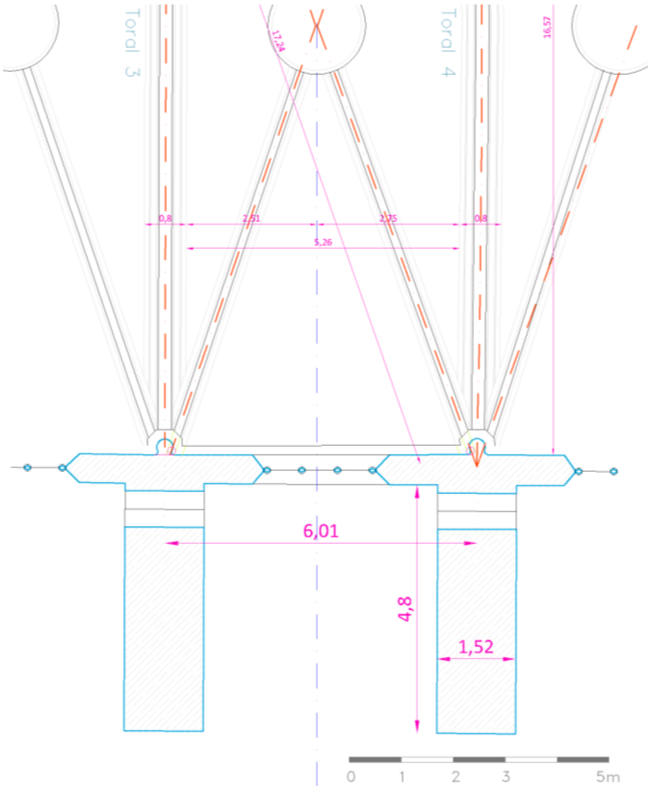



Figure 112 - massive and stiff side abutments

- Another observation related to the side wall and the abutment, which was also documented in the church survey, is the inclination of this side wall. This wall is inclined by an angle of 1.08° degrees to the exterior of the church and having a lateral displacement at the top of the wall of 0.52m. see figure 113. However, this displacement definitely happened during the construction of the church and the evidence is that the transversal vault reaches the side wall and covers this distance. A possible explanation to this inclination is that it occurred during the construction due to the lateral pressure of the cross vaults and the infill materials while the buttress was not constructed yet.

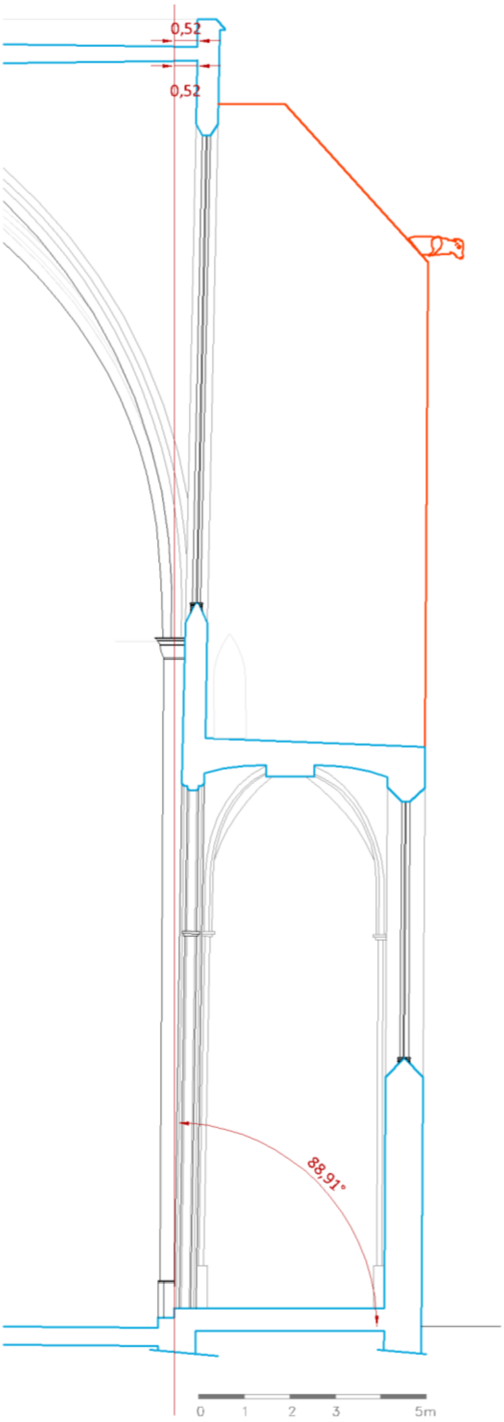
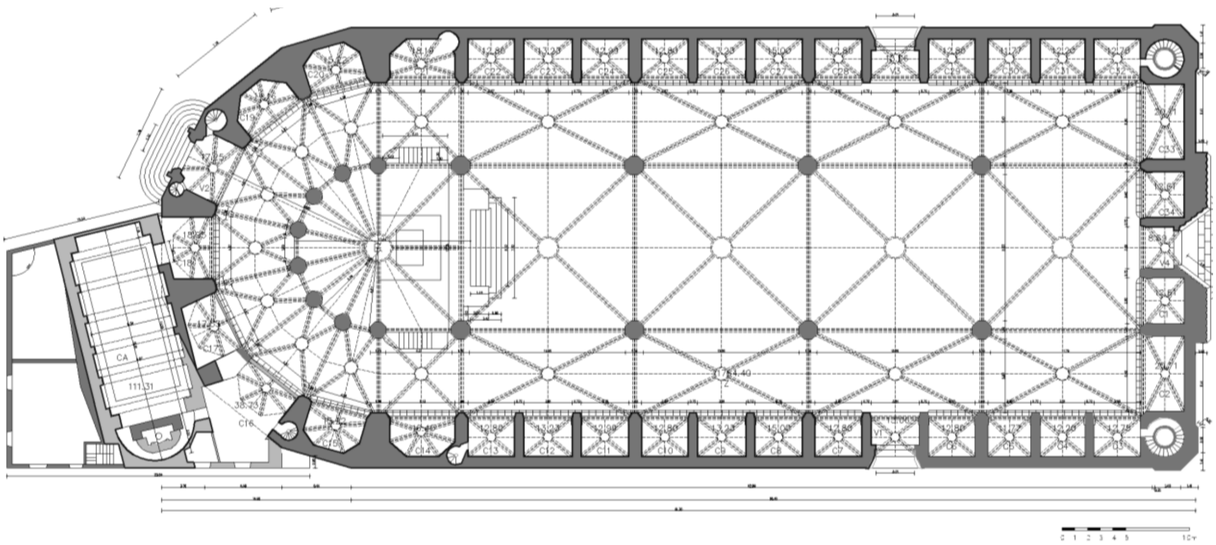


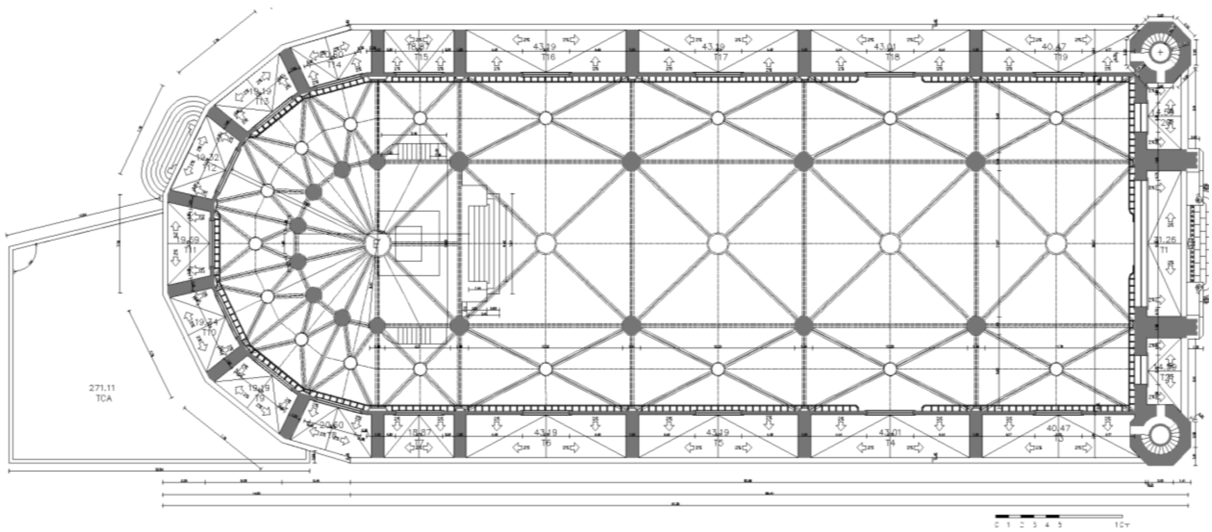

Figure 113 - recorded inclination of the side wall and abutment

## 5.3 Santa Maria Del Mar Cathedral, Barcelona:

### *5.3.1 General description of the church:*

The cathedral of Santa Maria del Mar is a beautiful and important 14th-century church located in Barcelona city. It is considered one of the finest and most complete examples of the Catalan Gothic architecture. It includes some interesting structural innovations shared only by the Barcelona and Mallorca cathedrals. The cathedral is also characterized by its huge clear space due to the large spans and height of the vaults and the slenderness of its middle nave columns.





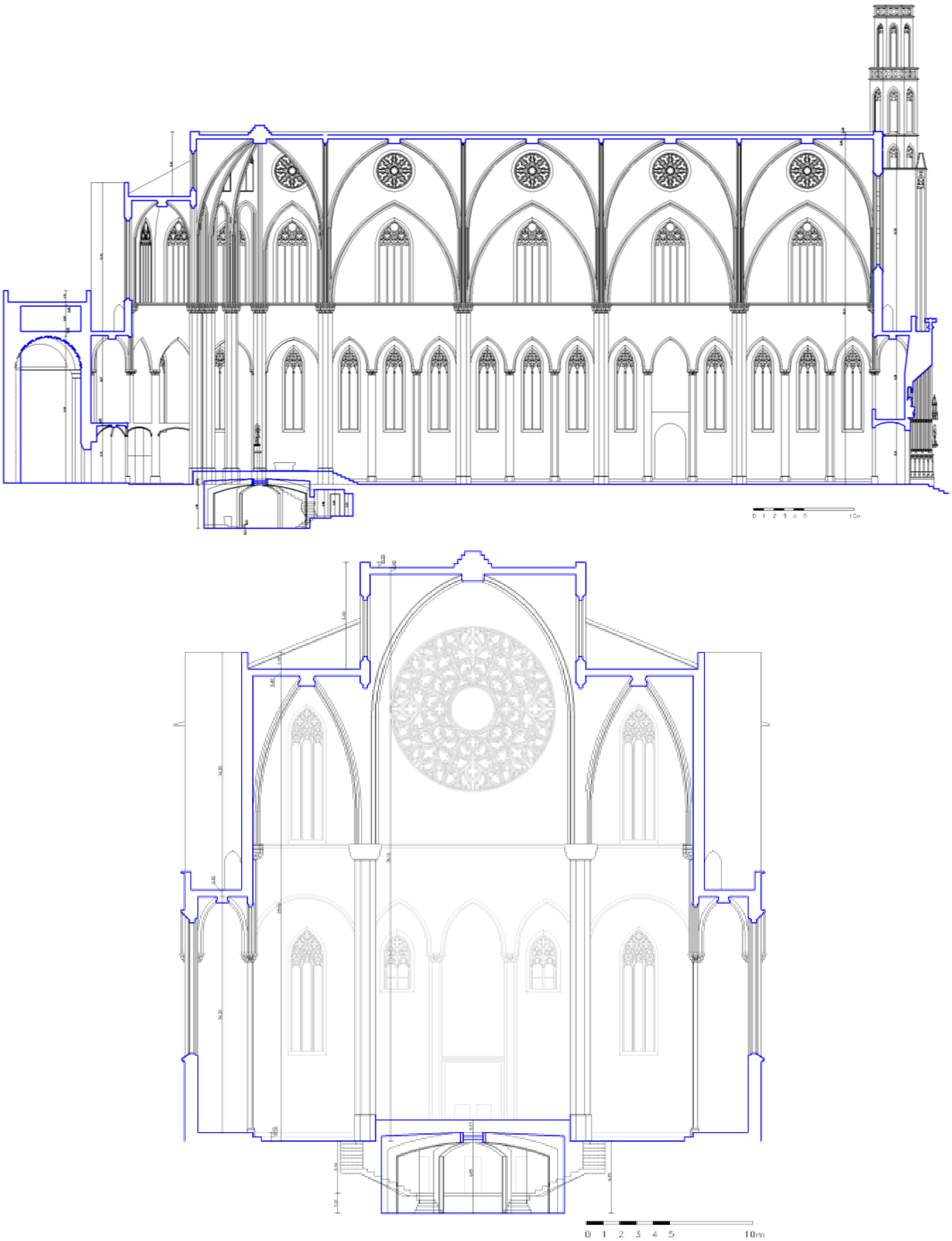


Figure 114 - Architectural drawings of the SMM church (plans, Long. and Trans. sections).

The building is composed of three naves, formed by four sections and an altar that consists of a polygon of seven sides, all covered by cross-vaults. The octagonal columns have a span of 13 meters from each other, forming four central sections of 13 by 13 meters. The Catalan gothic style uses buttresses instead of flying arches and the spaces between the end of the lateral naves and the end of the buttresses are included inside the structure, using them to locate little chapels. Figure 116 show the plans, longitudinal and transversal sections of the cathedral.

Figure 115 Interior of the church of SM del Mar

The interior of the church is distinguished by a clear volume creating almost a single space, a characteristic of the Southern Gothic architecture, with a preference for the longitudinal orientation without transept, as shown in figure 117. The church is composed by three naves with 13x13m square sections in the central nave and rectangular sections in the lateral naves. The altar consists on a polygon of seven sides. Little chapels are found in the space created between buttresses.

### *5.3.2 The Geometry of the Vault:*

The squared sections of the main nave are covered by cross vaults surrounded by diaphragmatic arches at a height of 32m .The lateral vaults have a significant height compared to the central one (26m) and are also covered by rectangular cross vaults (figure 118). The altar of the church is covered with a vault with palm shape and has a big circular keystone in the centre (2m diameter).

Figure 116 Picture showing the repetitive cross vaults

The same procedures early described in the analysis of the intersected vaults of Santa Maria del Pi church also implies for this second case study. I started by investigating the geometry of the vault and its constituting elements. I defined the dimensions for each of them and I assumed the same mechanical characteristics as their specific weights. Then again I used the Sketchup software to construct the model as shown in figures 119.

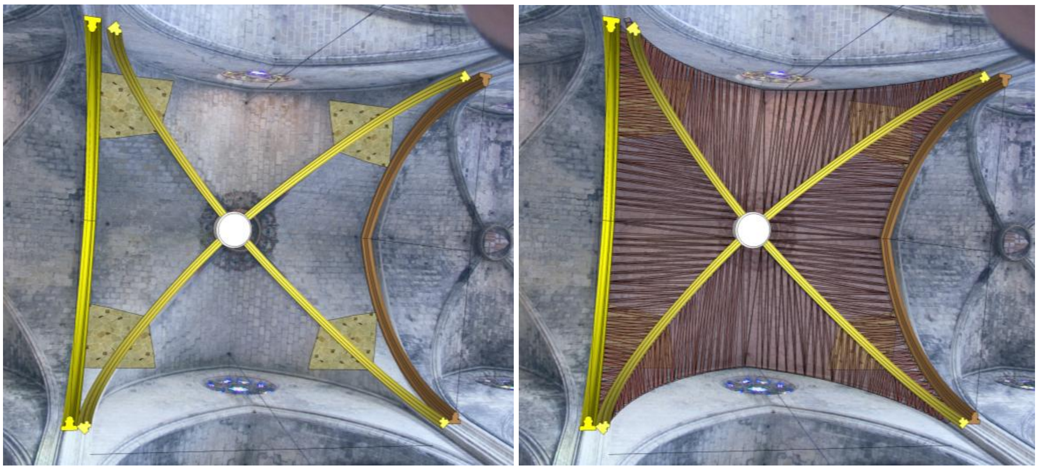


Figure 117 Schematic drawing of SMM Model

### 5.3.3 *Analysis of the cross sections:*

The analysis of the vault of Santa Maria del Pi proved that the parallel decomposition according to Beranek is not possible as the cross crib cannot support the combination of loads following this decomposition. Thus, in my analysis for this second case, I chose to do my analysis only for the diagonal decomposition according to Mark. The fact that this vault has an almost squared plan can influence the analysis results. Also, the analysis can clarify the previous results and can explain them.

I defined a series of diagonal sections slicing the symmetrical vault, having an interval of 1.15 m on the ridge and they are 5 sections named from MM-D-sec1 to MM-D-sec5 as shown in figure 120. It is to be cleared that width of each section varied between 1.14 to 0.86 m at the middle of the vault.

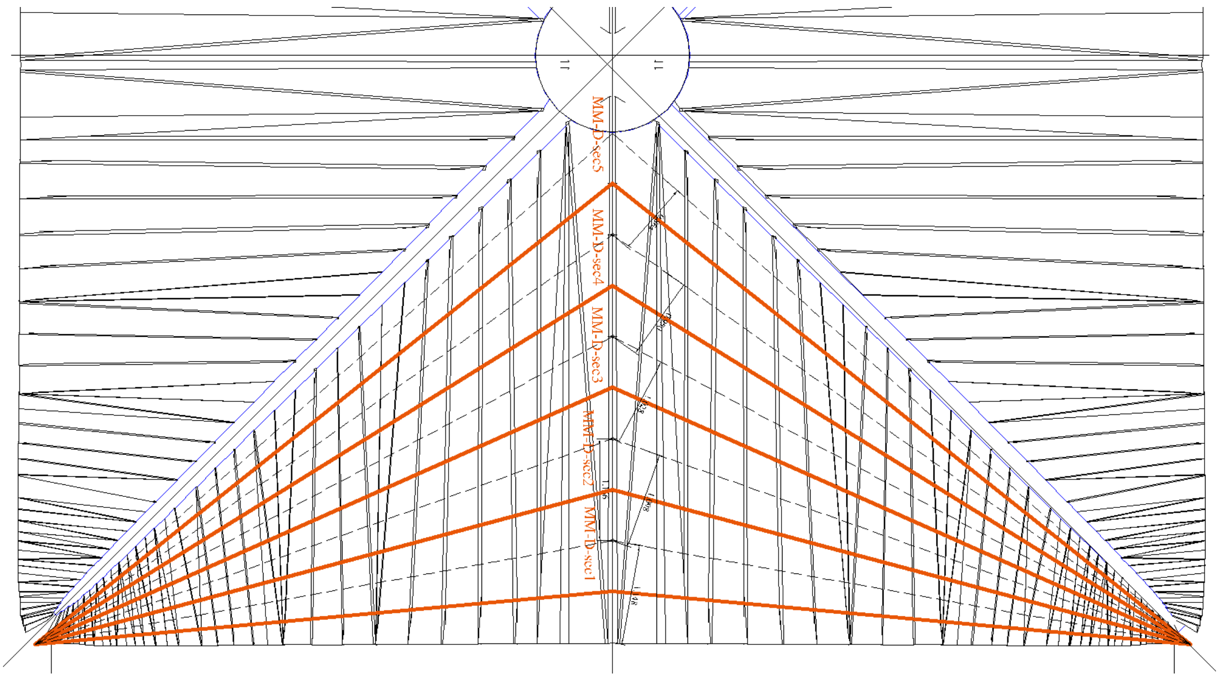



Figure 118 slicing sections of the SMM vault

The analysed sections, such as SMM-D-sec1 shown in figures 121 and 122 confirm that the developed line of thrust exists within most of the stone voussoirs of the arch and then goes outside to the sound infill near the lower part of the arch.

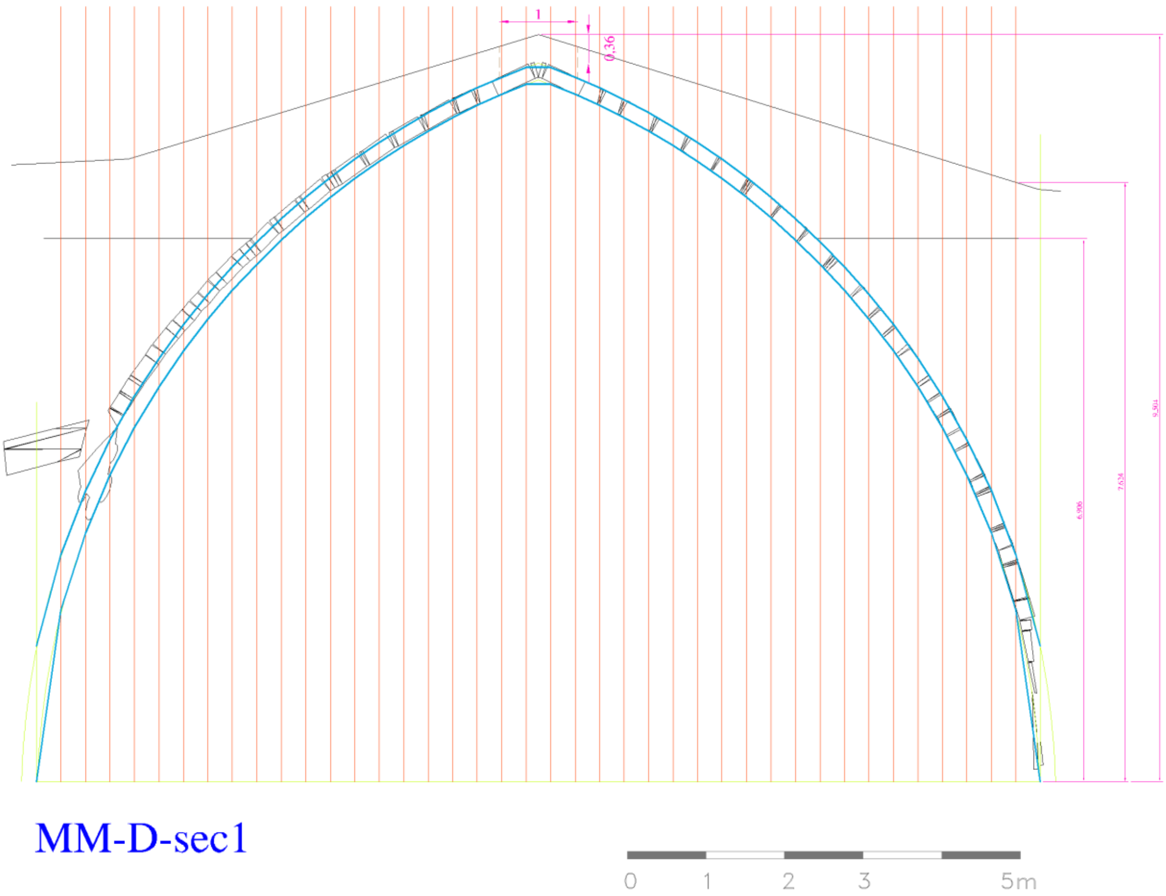



Figure 119 Diag-sec1 CAD geometry and dimensions.

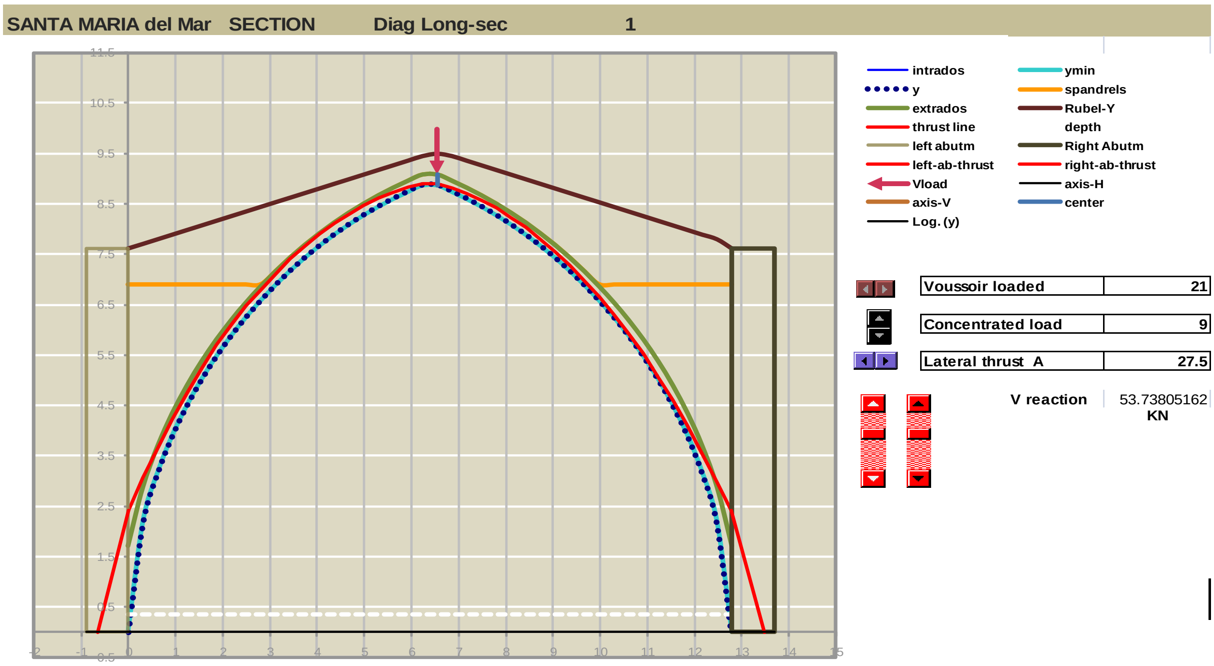



Figure 120 graphical analysis of SMM diag-sec1

First, I analysed the cross rib due to its own weight and the weights of the above infill layers. I obtained the same results as for the diagonal sections which imply that the line of thrust is developing through the arc voussoirs and goes to the sound adjacent infill in the spandrel area. This is shown in figures 123-124.

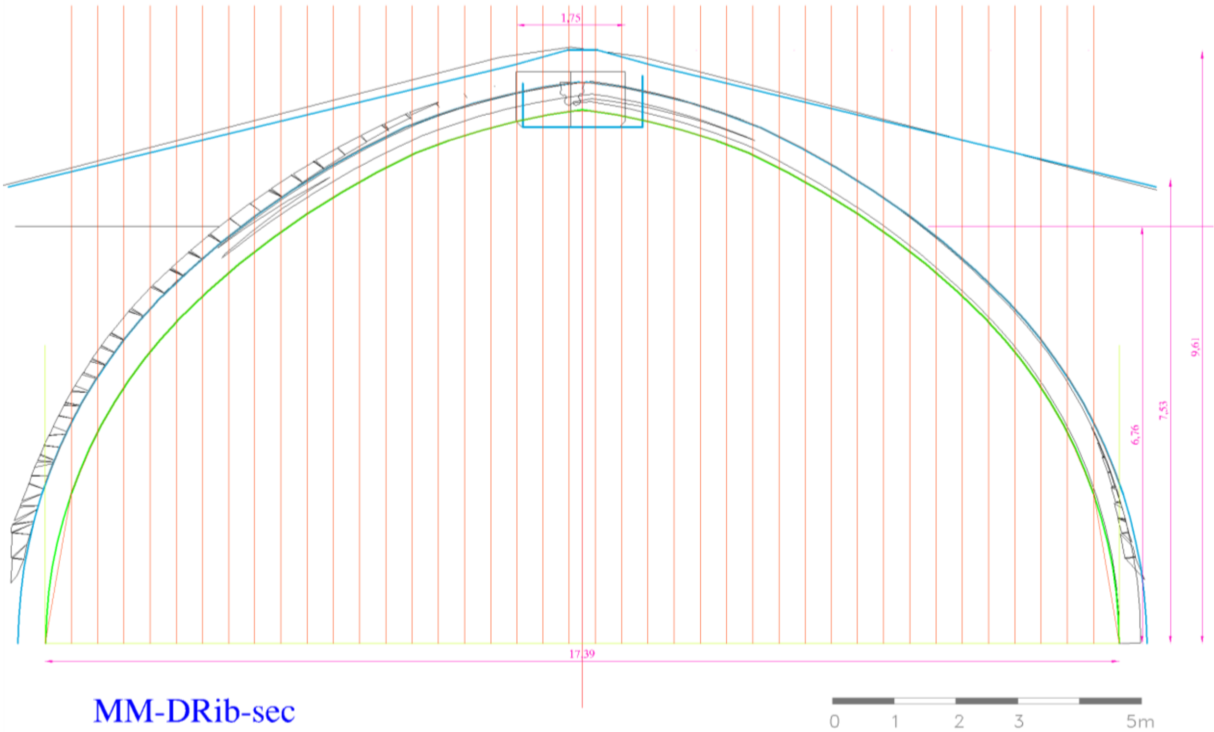



Figure 121 cross rib section of SMM church

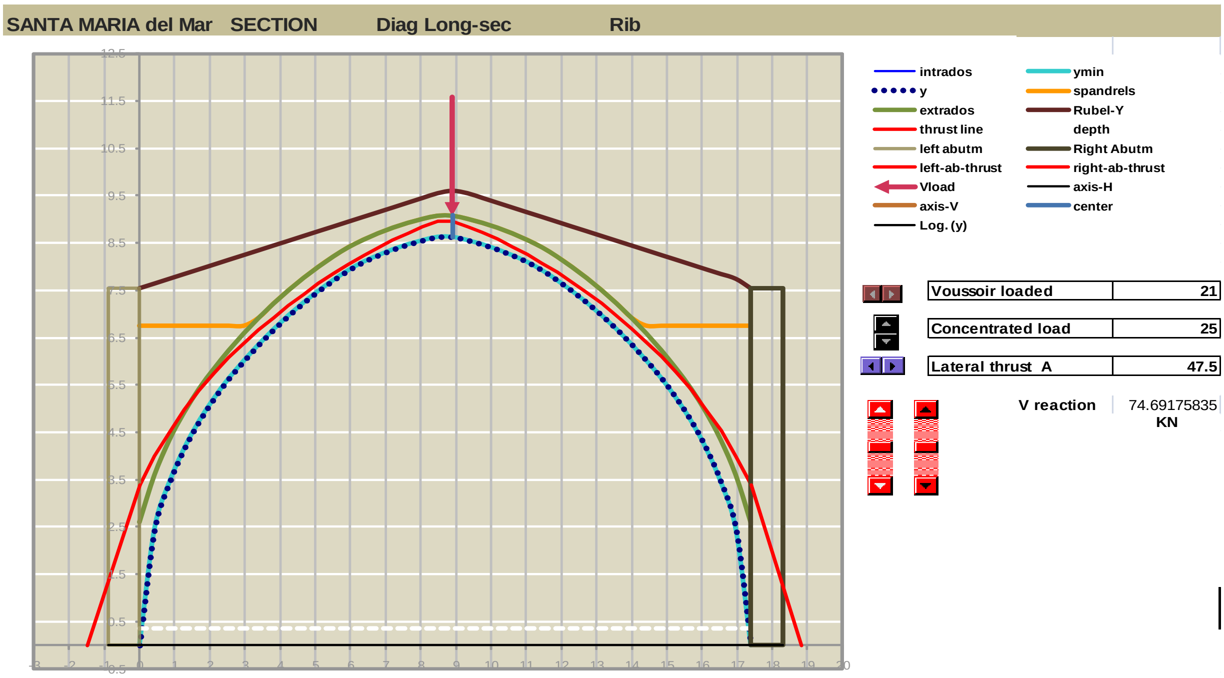



Figure 122 analysis of SMM cross rib section

Lately, I reanalyzed the cross rib by only applying its own weight thus neglecting the weights of both light and sound infill layers. This time I obtained a line of thrust going within the whole section of the stone voussoirs. (See figure 125)

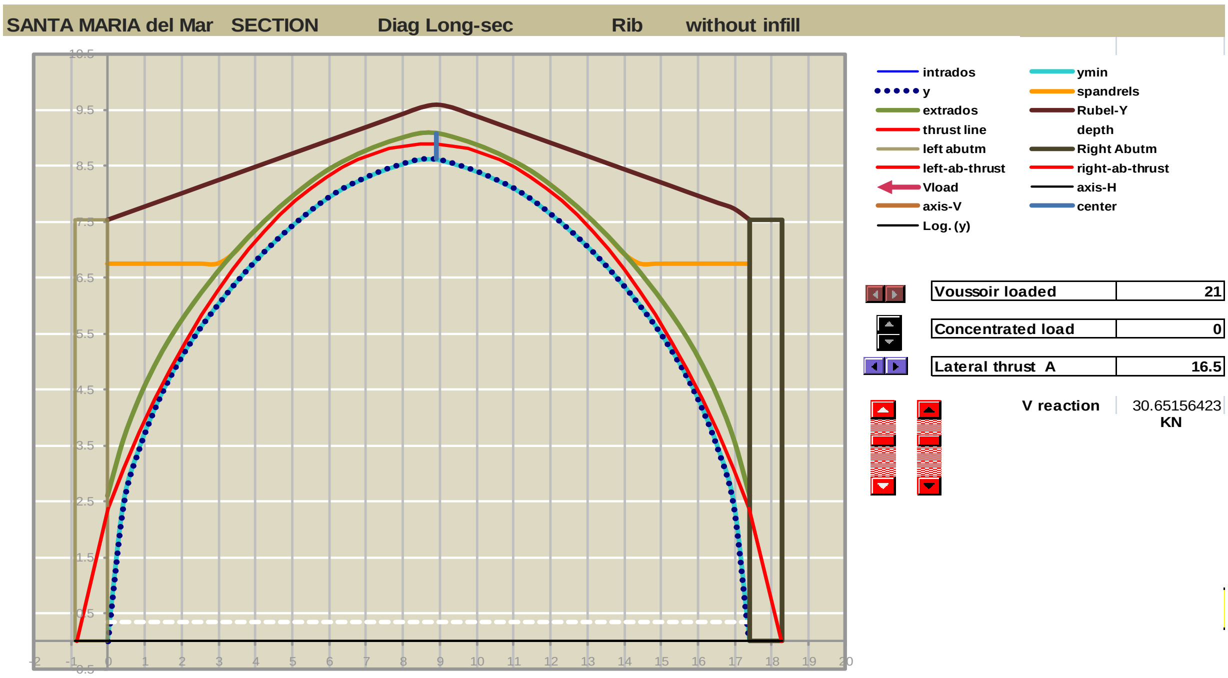



Figure 123 analysis of SMM cross rib without infill weight

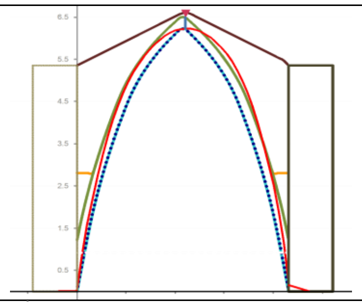
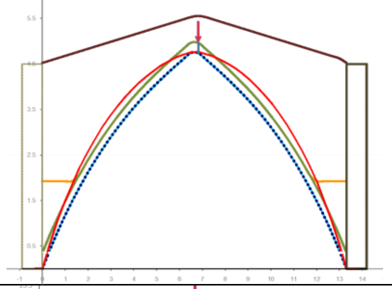
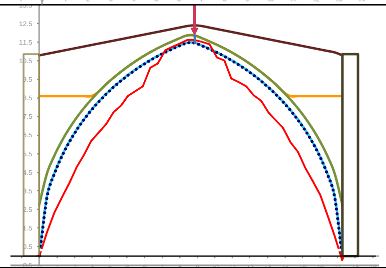
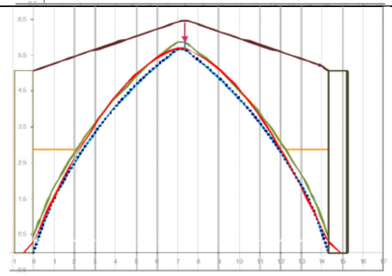
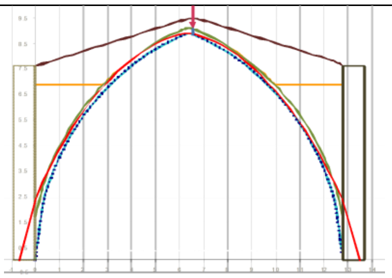
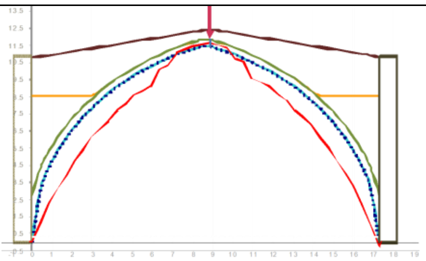
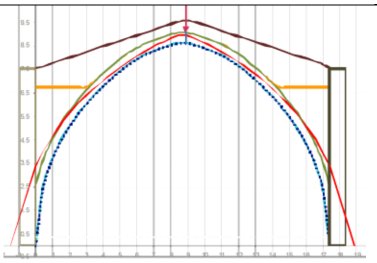
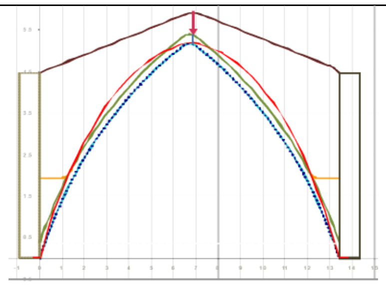
### 5.3.4 *Results:*

1- The stone pointed arch of the cross rib supports its own weight and develops a line of thrust within its cross section. A structural state of equilibrium is clear and validated by the graphical analysis.

2- The diagonal cross section of the intersected pointed ribbed vault develops a line of thrust that goes outside the section into the sound infill when it is loaded by the infill layers and the roofing tiles. This implies that spandrel sound infill act with the stone voussoirs as a composite section and transfers the lateral stresses to the adjacent structural elements such as abutments and buttresses.

3- The attained values for the minimum and maximum horizontal thrusts are almost identical for the analyzed arches.

Table 1 comparison of analysis of the arches

| | SM del Pi | SM del MAR |
|---|---|---|
| **Transversal vault PARALLEL Beranek decomposition** |  | |
| **Long vault PARALLEL Beranek decomposition** |  | |
| **Cross rib PARALLEL** |  | |
| **Long Vault DIAGONAL Mark decomposition** |  |  |
| **Cross rib DIAGONAL Mark decomposition** |  |  |
| **Long Vault New Diagonal Researcher decomposition** |  | |

# 6. CONCLUSIONS:

## 6.1 Regarding the form and geometry of the vaults:

- The survey and documentation of the structure is very important and crucial for the understanding and the analysis of the historical structures. In the case of the Gothic vaults, it should be more sensitive as the different components of the vault vary in their shapes and their dimensions. Capturing the exact geometry, that is not an easy task, will support an almost exact graphical representation needed for the drawings or the model construction thus the analysis.
- The tools and techniques used for the survey are still not fully efficient for capturing and interpreting the geometry. It is not possible to rely on just one tool as laser scanners or photogrammetry. Only the integration of different techniques can do much help to overcome this issue.
- Another difficulty related to the vaults, which can be called the inner or hidden geometry, is to define the different layers of the components and their dimensions; such as the infill. They are visually hidden and not easily reached. Digging and destructive testing are highly not recommended and many times forbidden.
- The general lack of information regarding the Gothic and a historic structure, due to its ambiguity, makes necessary the suggestion of realistic and coherent assumptions for some values to undertake the structural analysis.
- The lack of information about the used materials, their constituents and their mechanical properties is likely one of the main challenges dealing with historical structures. Exact values of the specific weight or relative density are not usually accurate and sometimes assumed.
- The site visits and the visual inspection are very important for understanding the different parts and components of the structure.
- Investigating the construction procedures, if possible, can be helpful for the understanding of the structure and its elements.
- Recording of the deformations and the cracks of the structure is very important for the structural analysis.

## 6.2 Regarding the graphical analysis:

- The graphical analysis can be carried out for the historical masonry structures and specially the gothic vaults by implying the safe theorem of limit analysis following the assumptions of infinite compressive strength, friction resistance and zero tensile strength for the stone masonry.
- The graphical analysis is favored for the study and analysis of the masonry historical structures, specially the gothic vaults. It is simple, effective and economical in terms of time, effort and resources. Generally, it can determine the collapse mechanism and the load multiplier that is an indicator of the structural safety of the structure.
- The use of Excel spreadsheets facilitates the calculations and minimizes the duration of analysis especially in case of repetitions.
- A main difficulty when using this technique is the introduction of the actual shape and geometry of the elements to the spreadsheets. Succeeding in using a smart point by point technique, that was developed by a cad routine instead of using the circumferential and parabolic equation for the arches, provided a more realistic analysis with the exact shape.

## 6.3 Regarding the conducted analysis:

- The analysis of two different types of Catalan vaults; as a rectangular quadripartite vault (SM del Pi) and a squared quadripartite vault (SM del Mar), allowed the comparison of the results and a better understanding of the behavior that proved to be unlike for both vaults regarding some aspects.

The case of the vault of Santa Maria del Pi;

- The line of thrust was not achieved within the arches of the web of the vault and usually was going outside to the infill.
- The equilibrium status depends on the whole cross section meaning that the infill (light and sound) is acting in favor of transmitting the thrusts to the buttresses in the lateral direction.

The case of the vault of Santa Maria del Mar;

- The line of thrust was achieved within most of the stone arch portion and was going out at the bottom of the haunches to the outer sound infill.
- The equilibrium status depends on the stone arch and the sound infill only.

As for the similarities:

- The existence of a convincing line of thrust through the voussoir arches was not possible in all analyzed sections The vaults membranes composed of thin stones and having a pointed arch shape cannot support individually the combination of loads consisting of their own weights and the other loads applied to them. However, the structure is standing with no collapse which affirms a state of equilibrium. This can only be explained by the fact that the line of thrust goes outside of the arch to the exterior infill of the spandrel. This infill material has a reasonable compressive strength that can support the stresses developed by the arches of the vaults.
- For the pointed arches case, having a line of thrust passing by the sound infill (spandrel part) is in accordance to the approach undertaken by Abraham for the analysis of the pointed vaults were he started to analyse the arch from the level above the spandrel.
- Acording to the results, the decomposition of cross vault according to Mark is more realistic than the decomposition of Beranek. The path of forces acting in the vault would follow a diagonal path more than a parallel one. However, this doesn't mean that Mark's decomposition is the best approach as it may be possible that another type can exist. (remember that Mark used photoelasticity)
- The diagonal cross rib cannot support the loads applied by the webs of the vault as it was impossible to find a line of thrust passing through the rib when applying these loads. The cross section of the rib is also relatively small and is not capable of supporting such heavy loads. This result goes along with the theory that the cross rib has no a main structural bearing role.
- In almost all of the analysed thin arches of the vaults, the minimum and maximum values for the lateral thrust needed for the equilibrium of the arches were approximately the same. This

result is in accordance to the studies conducted by (Romano & Ochsendorf 2010) regarding the analysis of the pointed (Gothic) arch.

- The case of the pointed arch and its structural behavior is not exactly similar to the circular arch as it represents a special case where the behavior is quite singular and unique. It requires a more sensitive balance of forces, however it provides a higher capability of supporting vertical loads (both concentrated and distributed) with a much smaller thrust than the circular arch.

## 6.4 Recommendations:

- The ambiguity of the gothic vaults regarding the materials and its layers require more non destructive and minor destructive testing to clarify many issues and minimize the amount of assumptions with taken for the analysis of gothic vaults.
- The graphical static analysis is needed to understand the behavior of the historical structures, especially the gothic vaults. By implementing this analysis, we can understand the state of equilibrium for the structure and its behaviour. Later, the FEM modeling and other analysis techniques can then be used once the actual concepts are grasped and understood. Following this procedure, we can guarantee that we get realistic results.
- the graphical analysis is subjected to development, mainly according to computer applets and tools, however it would be very helpful if a 3d tool, which can allow the construction of the model combined with graphical analysis, is developed.

More research and analysis is needed for different examples and types of the Gothic vaults in order to reach a final conclusion about their stability. However, the results of this research can provide a good understanding about the problem and can contribute to the efforts undertaken for the conservation of such notable structural and architectural pattern.

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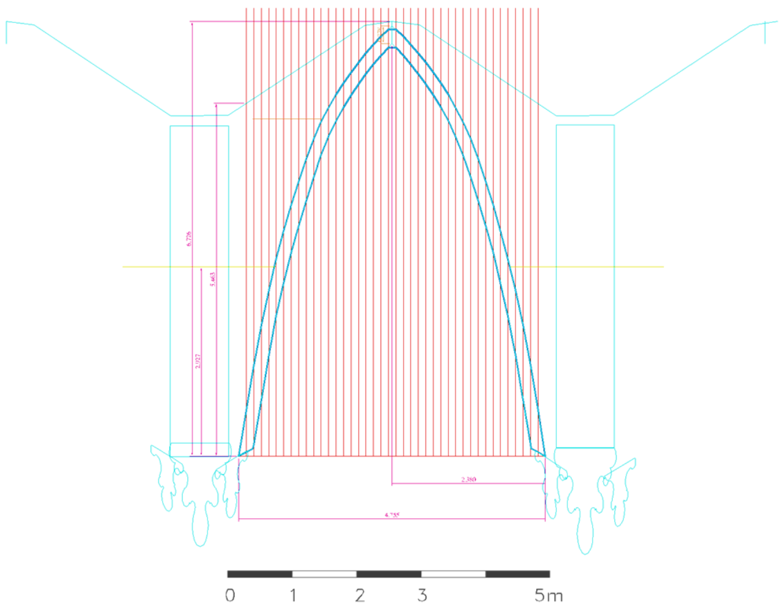
# 8. APPENDICES

## Appendix A

### The Excel Spreadsheet




| ARC with Spandrels | | Intrados | Arch extrados |
|---|---|---|---|
| Voussoir | x | y | ymax |
| | | | |
| 0 | 0 | 0 | 1.218256561 |
| 1 | 0.105261 | 0.631461077 | 1.80176057 |
| 2 | 0.210522 | 1.262061925 | 2.305247592 |
| 3 | 0.315783 | 1.83592561 | 2.757336414 |
| 4 | 0.421044 | 2.330639176 | 3.166652261 |
| 5 | 0.526305 | 2.775582511 | 3.543926415 |
| 6 | 0.631566 | 3.178290073 | 3.896269426 |
| 7 | 0.736827 | 3.551682377 | 4.227382863 |
| 8 | 0.842088 | 3.900361562 | 4.540255948 |
| 9 | 0.947349 | 4.227997527 | 4.818096509 |
| 10 | 1.05261 | 4.537553468 | 5.049818315 |
| 11 | 1.157871 | 4.800446523 | 5.254818994 |
| 12 | 1.263132 | 5.025147693 | 5.439853006 |
| 13 | 1.368393 | 5.223467825 | 5.608551915 |
| 14 | 1.473654 | 5.401577208 | 5.763658998 |
| 15 | 1.578915 | 5.564433296 | 5.907287939 |
| 16 | 1.684176 | 5.714305355 | 6.041096424 |
| 17 | 1.789437 | 5.852992721 | 6.166405541 |
| 18 | 1.894698 | 5.981941665 | 6.284283003 |
| 19 | 1.999959 | 6.102329006 | 6.393084596 |
| 20 | 2.10522 | 6.21512304 | 6.497363362 |
| 21 | 2.210481 | 6.21512304 | 6.497363342 |
| 22 | 2.315742 | 6.102329006 | 6.393084596 |
| 23 | 2.421002 | 5.981941665 | 6.284283003 |
| 24 | 2.526263 | 5.852992721 | 6.166405541 |
| 25 | 2.631524 | 5.714305355 | 6.041096424 |
| 26 | 2.736785 | 5.564433296 | 5.907287939 |
| 27 | 2.842046 | 5.401577208 | 5.763658998 |
| 28 | 2.947307 | 5.223467825 | 5.608551915 |
| 29 | 3.052568 | 5.025147693 | 5.439853006 |
| 30 | 3.157829 | 4.800446523 | 5.254818994 |
| 31 | 3.26309 | 4.537553468 | 5.049818315 |
| 32 | 3.368351 | 4.227997527 | 4.818096509 |
| 33 | 3.473612 | 3.900361562 | 4.540255948 |
| 34 | 3.578873 | 3.551682377 | 4.227382863 |
| 35 | 3.684134 | 3.178290073 | 3.896269426 |
| 36 | 3.789395 | 2.775582511 | 3.543926415 |
| 37 | 3.894656 | 2.330639176 | 3.166652261 |
| 38 | 3.999917 | 1.83592561 | 2.757336414 |
| 39 | 4.105178 | 1.262061925 | 2.305247592 |
| 40 | 4.210439 | 0.631461077 | 1.80176057 |
| 41 | 4.3157 | 0 | 1.218256561 |

Exportation of the ring coordinates to excel.

The defined equations for the calculation of the weights of the arch voussoirs using the Excel spreadsheets, is the following:

((H52+H51-J52-J51)/2*g+ (I52+I51-H52-H51)/2*gspan+ (K52-I52)*greb)*(C52-C51)*width

*Where (H52+H51-J52-J51)/2 is height of stone part (m)*

*(g)= specific weight of stone (KN/$m^3$)*

*(I52+I51-H52-H51)/2 is height of sound infill (spandrel) part (m)*

*(gspan) = specific weight of sound infill (KN/$m^3$)*

*(K52-I52) is height of light infill (m)*

*(greb) = specific weight of light infill (KN/$m^3$)*

*(C52-C51) is the width of voussoir and the portion (m)*

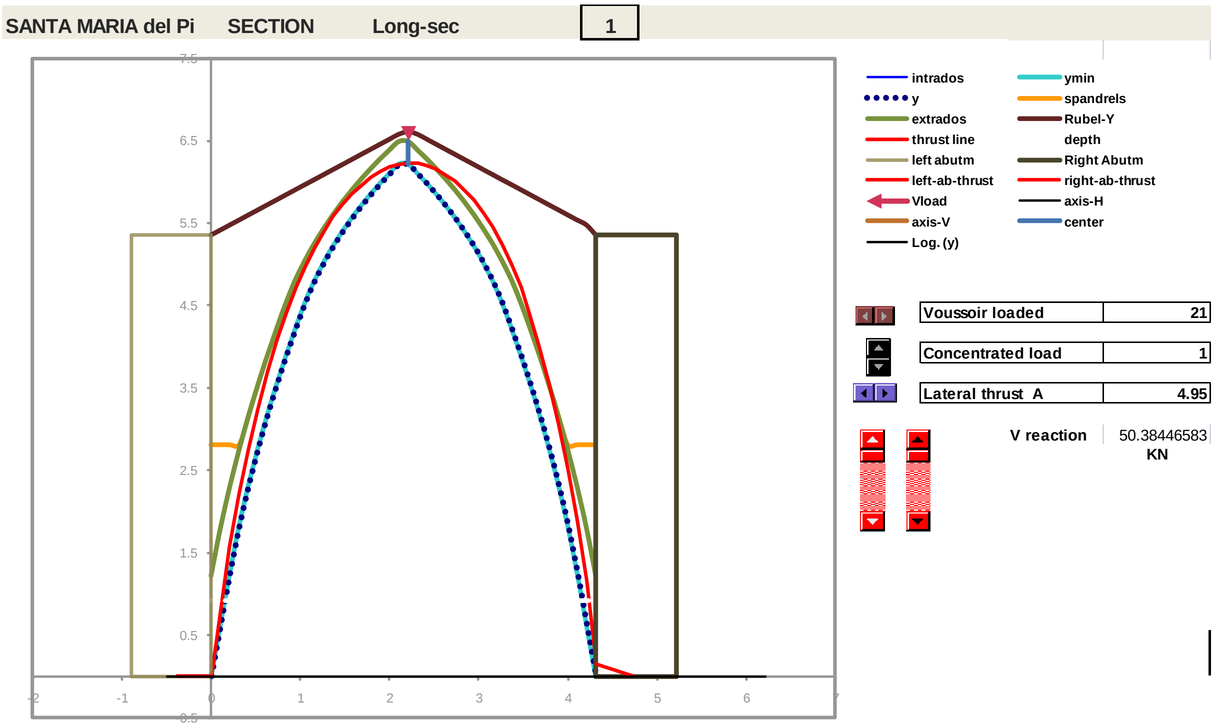
*(Width) is the thickness of the analyzed cross section (m)*

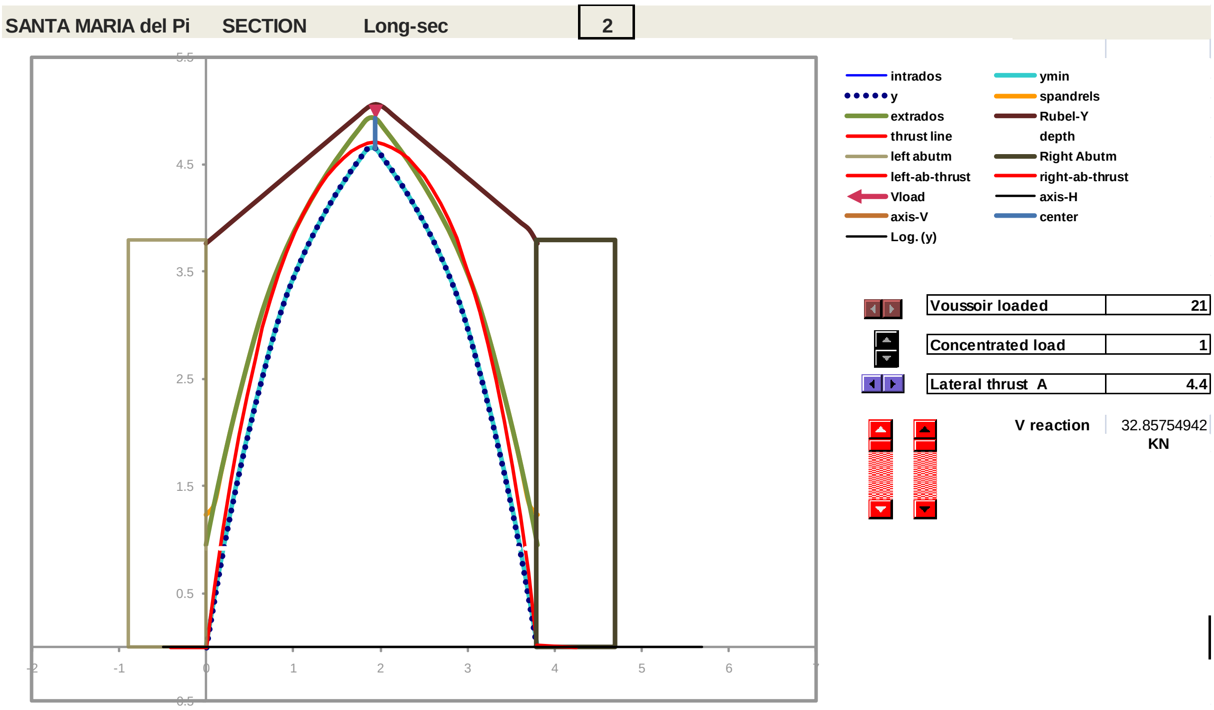
The sheet includes different cells for applying a concentrated vertical load on the top ring of the extrados, and which can be moved to any voussoir. In addition, also different horizontal and vertical loads can be defined and applied for each voussoir. A global horizontal force is also applied to the system and can easily be varied. This force is crucial to achieve equilibrium status for the system and statically analyze it.

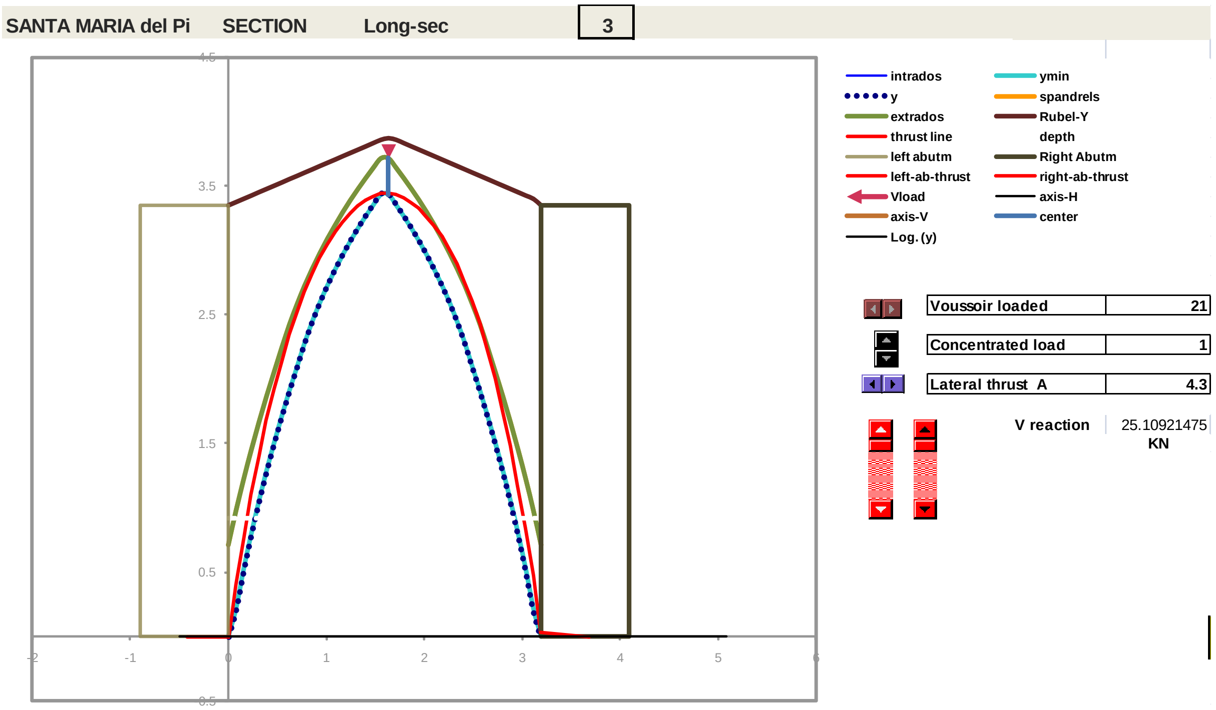
- The next step is the calculation of the moment for the resultant of weights, located at the middle of the voussoir, to the end point of the span.
- Later, I calculate moment due to the lateral and vertical applied loads also to the end point of the span.
- Before reaching the last calculation step, for each voussoir horizontal coordinate, I define two columns for the calculation of the change in the total moment value and also for the change of the horizontal load (force)
- The last calculation step is for defining the line of thrust. By using the equilibrium equation for each voussoir, I calculate a vertical coordinate at which the forces and moments acting on the designated voussoir are in equilibrium and this vertical coordinate is the position of the lateral reaction needed for the equilibrium.

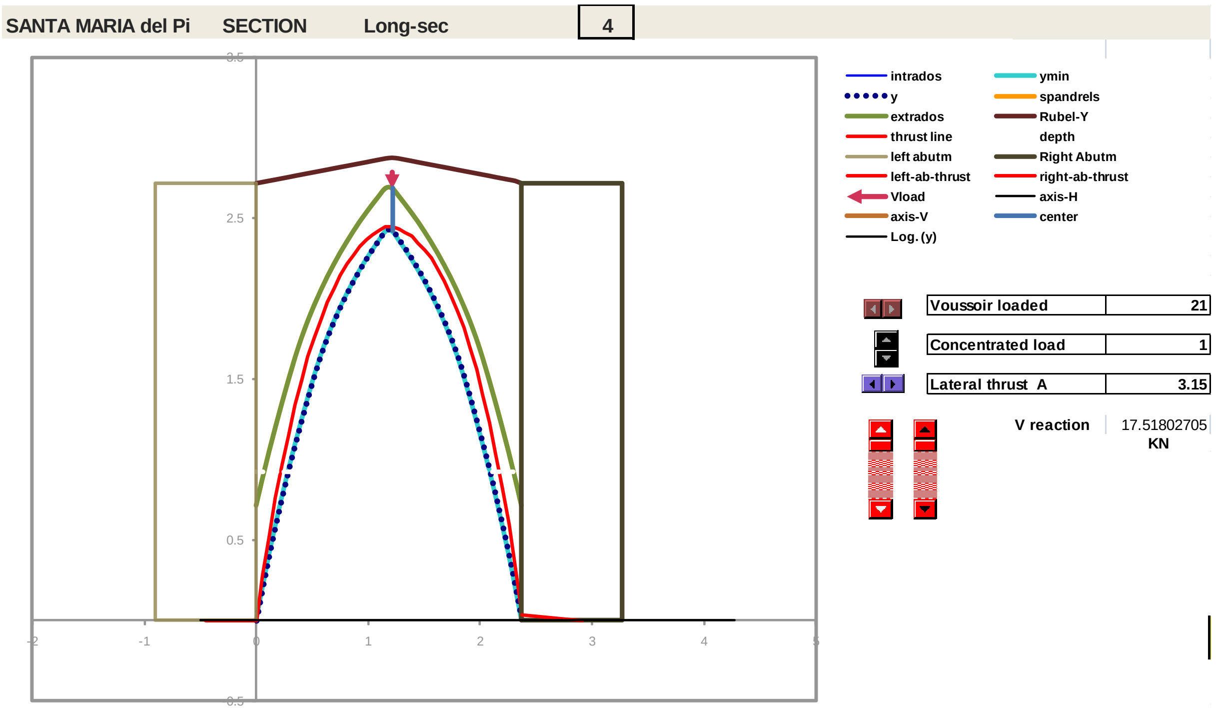
# Appendix B

## Santa Maria del Pi- Transversal Vault- Longitudinal sections




SANTA MARIA del Pi
SECTION
Long-sec
3
intrados
y
extrados
thrust line
left abutm
left-ab-thrust
Vload
axis-V
Log. (y)
ymin
spandrels
Rubel-Y
depth
Right Abutm
right-ab-thrust
axis-H
center
Voussoir loaded 21
Concentrated load 1
Lateral thrust A 4.3
V reaction 25.10921475
KN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Long-sec
4
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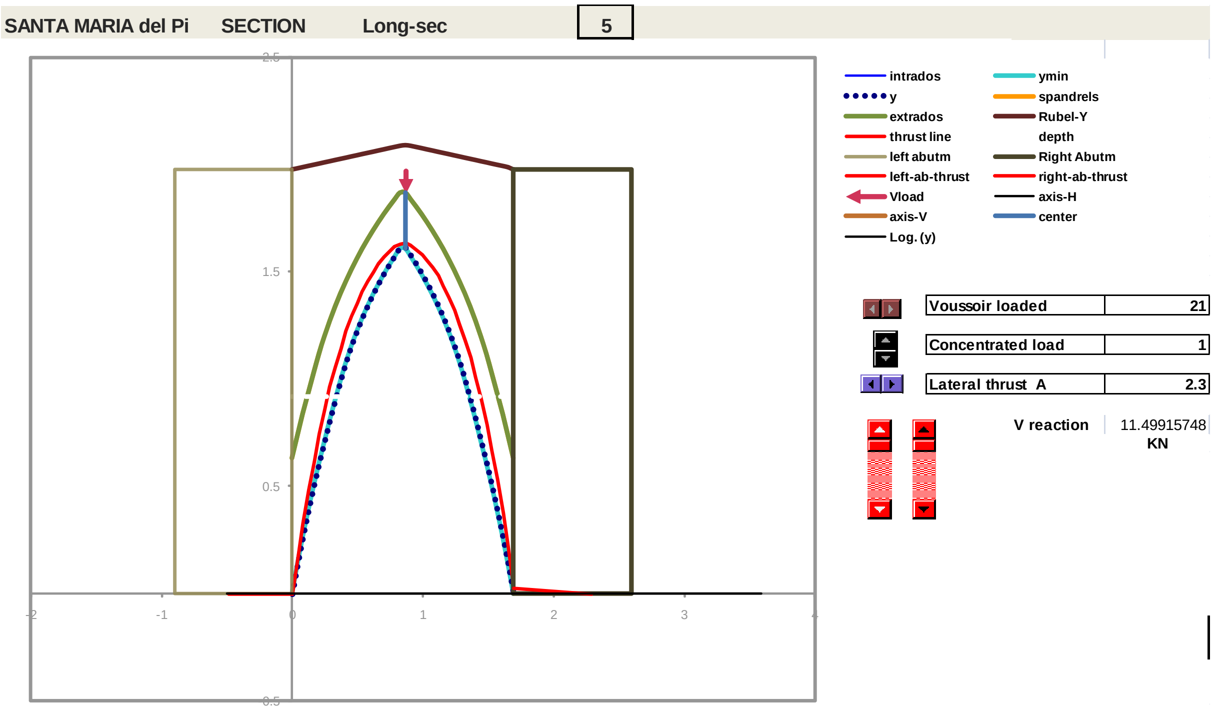
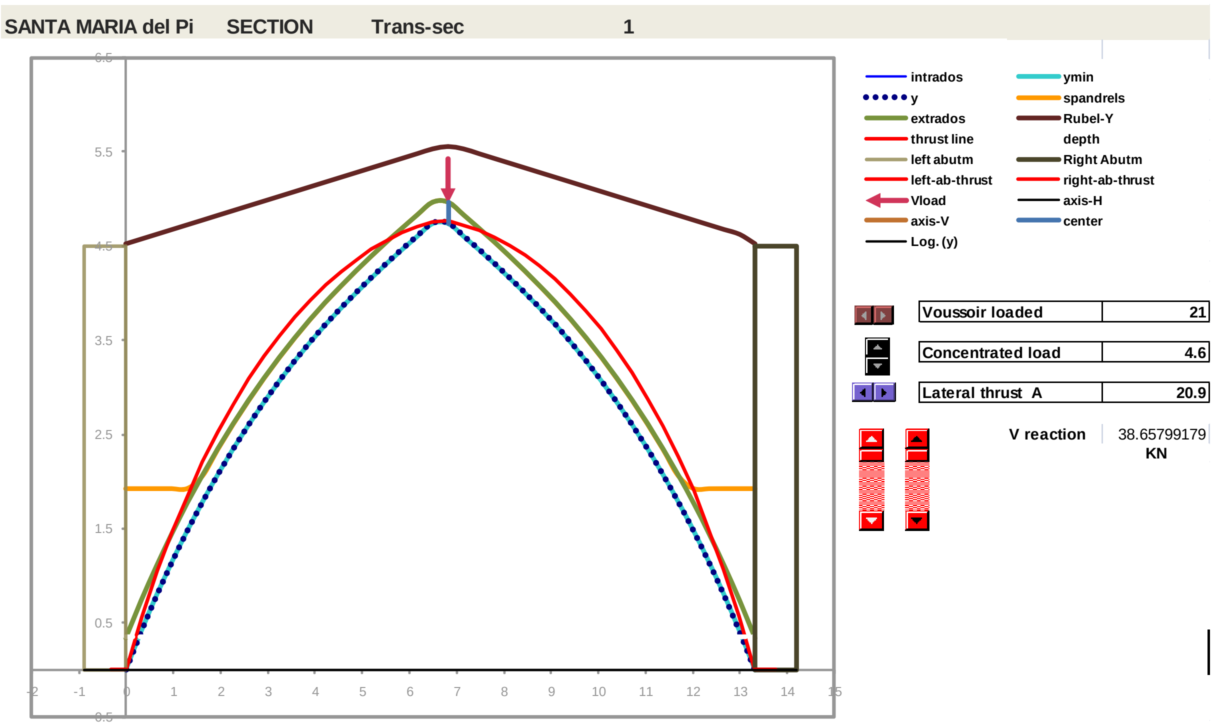
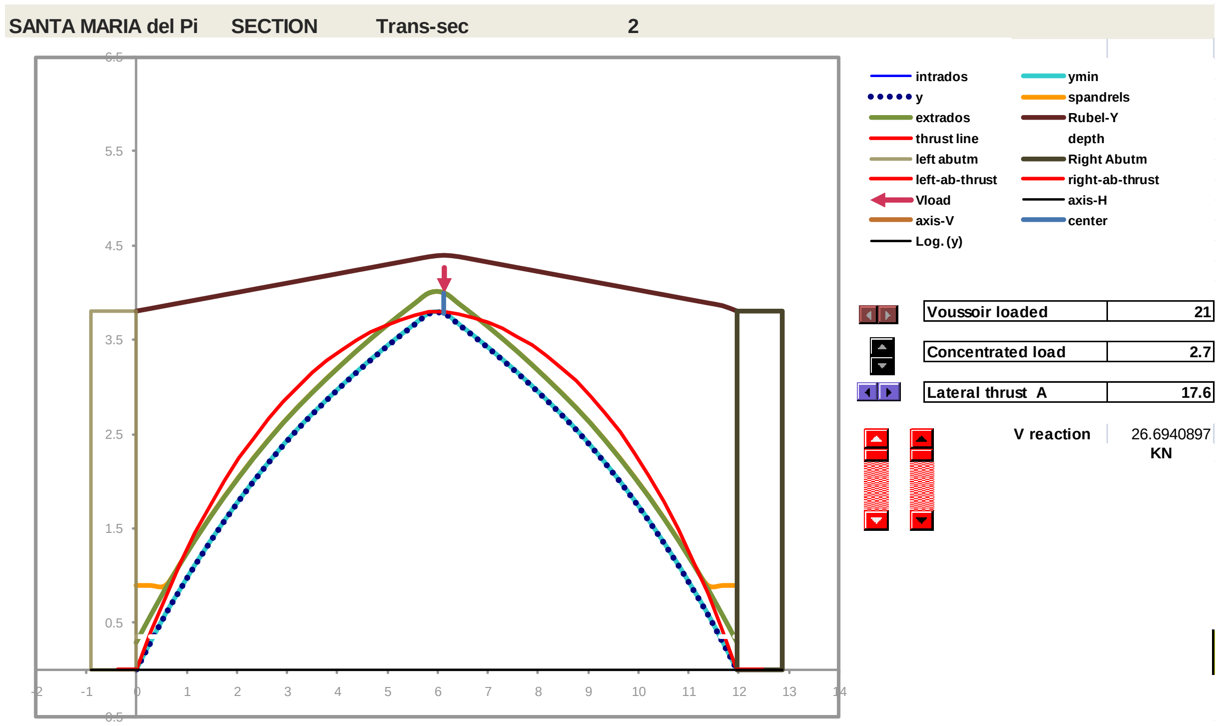
y
extrados
thrust line
left abutm
left-ab-thrust
Vload
axis-V
Log. (y)
ymin
spandrels
Rubel-Y
depth
Right Abutm
right-ab-thrust
axis-H
center
Voussoir loaded 21
Concentrated load 1
Lateral thrust A 3.15
V reaction 17.51802705
KN




## Santa Maria del Pi- Longitudinal Vault- transversal sections

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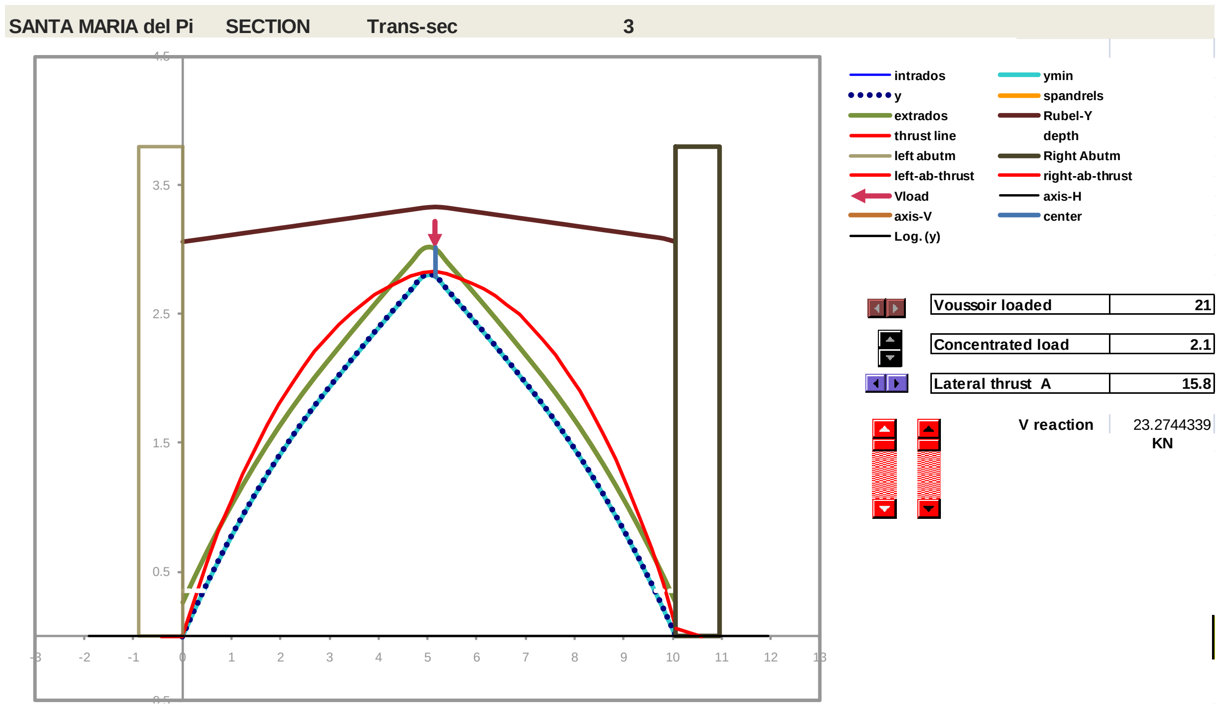
y
extrados
thrust line
left abutm
left-ab-thrust
Vload
axis-V
Log. (y)
ymin
spandrels
Rubel-Y
depth
Right Abutm
right-ab-thrust
axis-H
center
Voussoir loaded 21
Concentrated load 2.7
Lateral thrust A 17.6
V reaction 26.6940897
KN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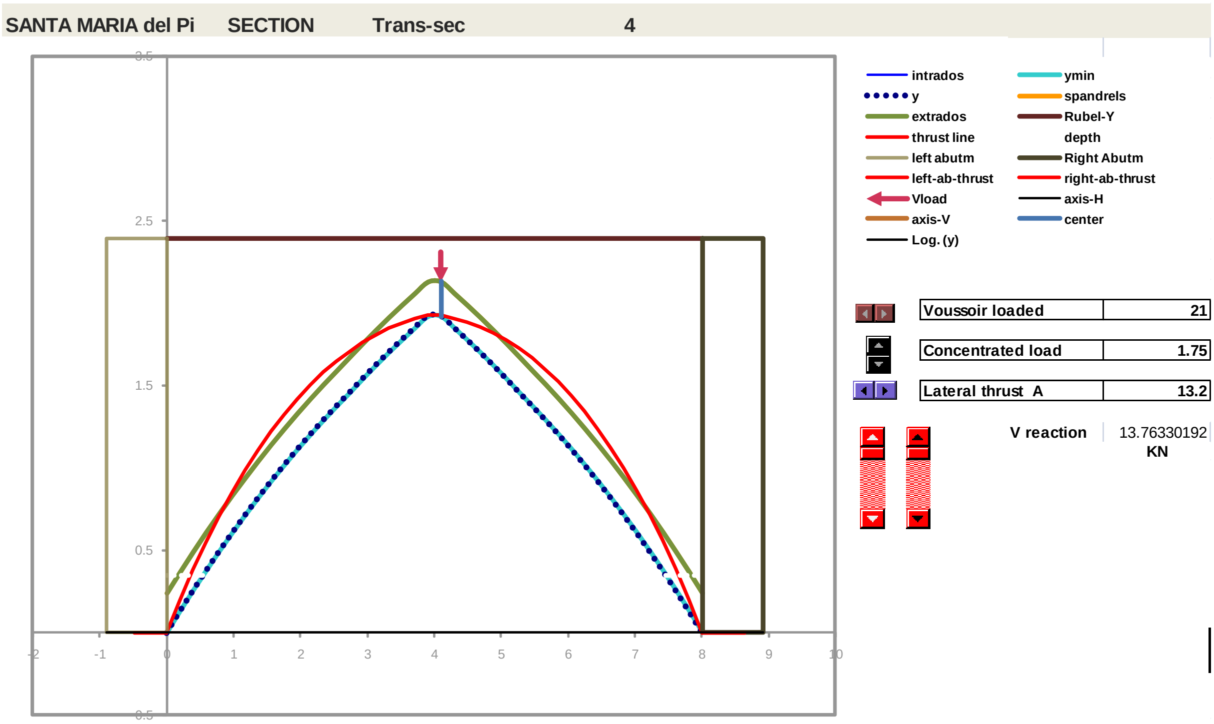
y
extrados
thrust line
left abutm
left-ab-thrust
Vload
axis-V
Log. (y)
ymin
spandrels
Rubel-Y
depth
Right Abutm
right-ab-thrust
axis-H
center
Voussoir loaded 21
Concentrated load 2.1
Lateral thrust A 15.8
V reaction 23.2744339
KN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y
extrados
thrust line
left abutm
left-ab-thrust
Vload
axis-V
Log. (y)
ymin
spandrels
Rubel-Y
depth
Right Abutm
right-ab-thrust
axis-H
center
Voussoir loaded 21
Concentrated load 1.75
Lateral thrust A 13.2
V reaction 13.76330192 KN
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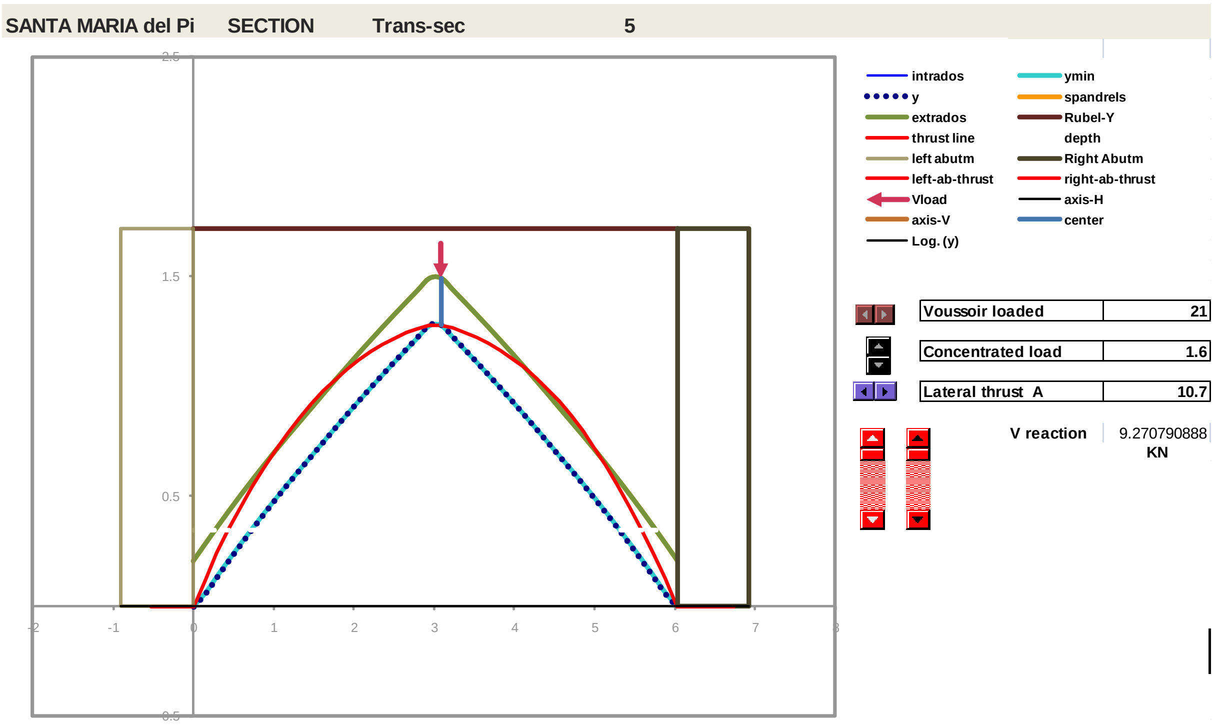
1.5
0.5
-1
0
1
2
3
4
5
6
7
8
9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y
extrados
thrust line
left abutm
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Vload
axis-V
Log. (y)
ymin
spandrels
Rubel-Y
depth
Right Abutm
right-ab-thrust
axis-H
center
Voussoir loaded 21
Concentrated load 1.6
Lateral thrust A 10.7
V reaction 9.270790888 KN
1.5
0.5
-1
0
1
2
3
4
5
6
7

## Santa Maria del Pi- Transversal Vault- Diagonal sections

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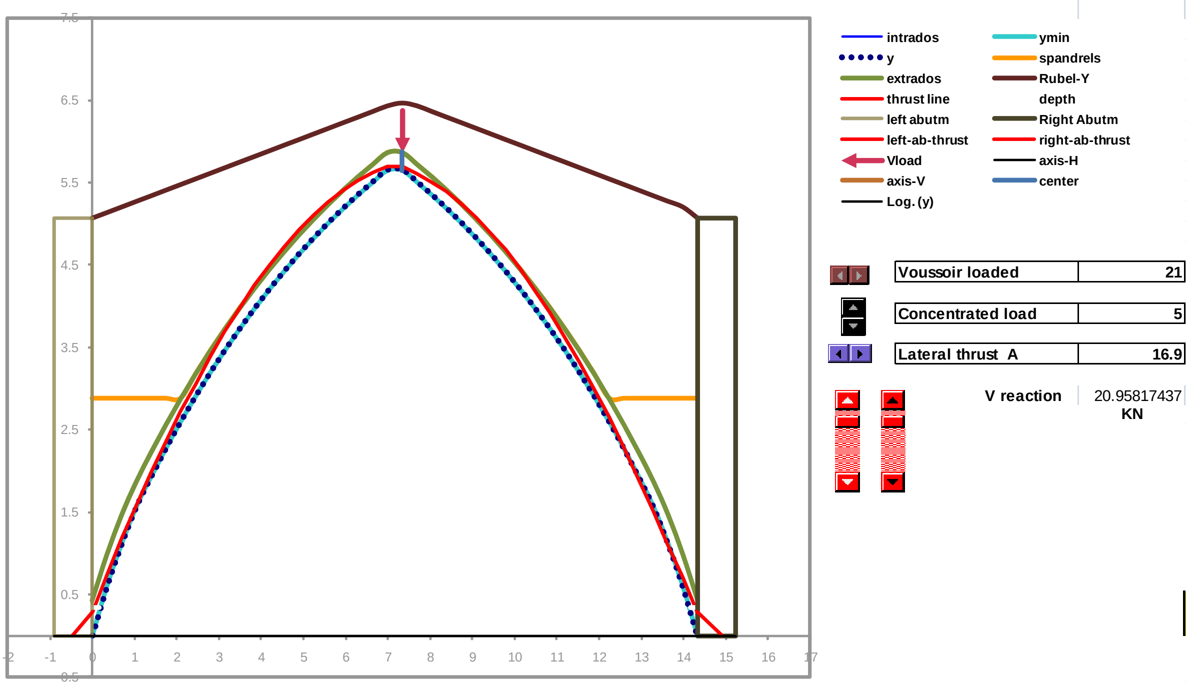
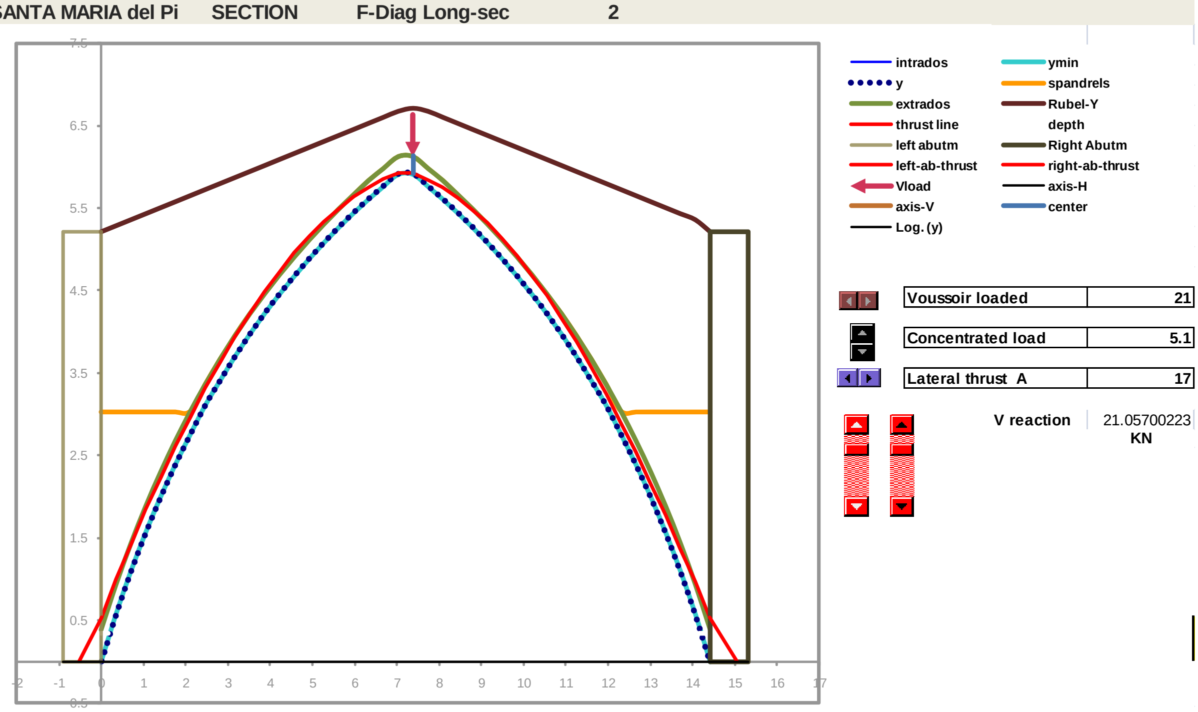
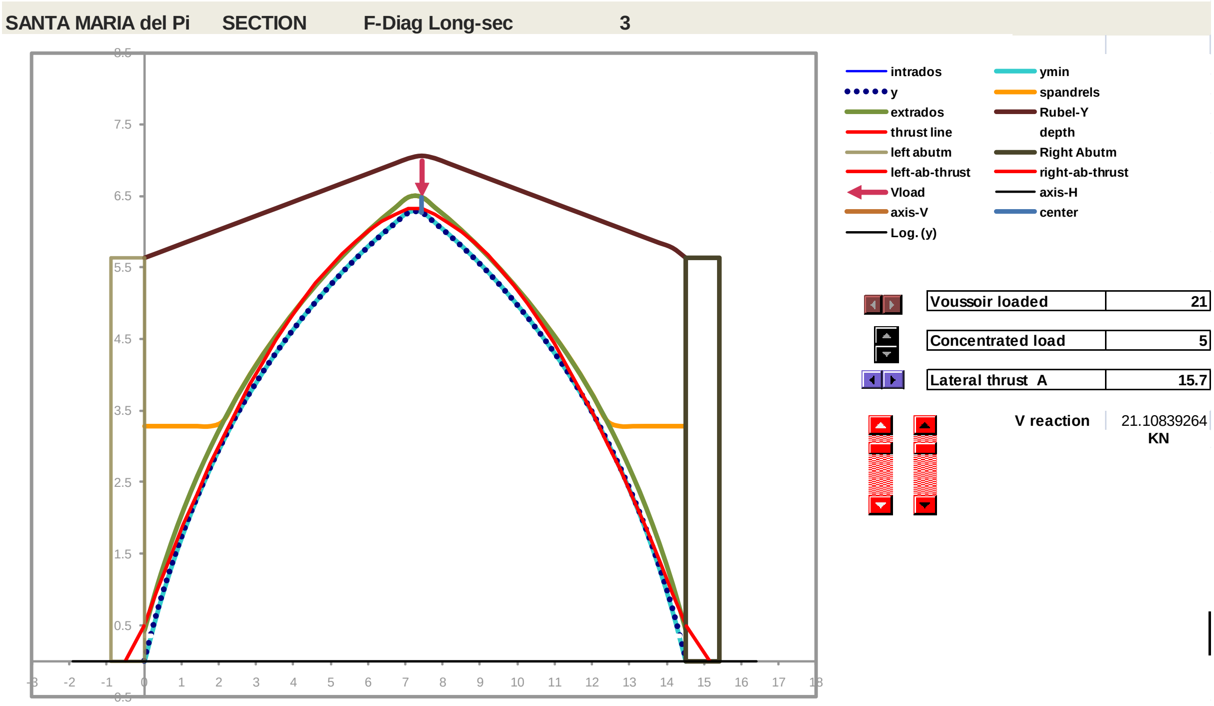



SANTA MARIA del Pi SECTION F-Diag Long-sec 2

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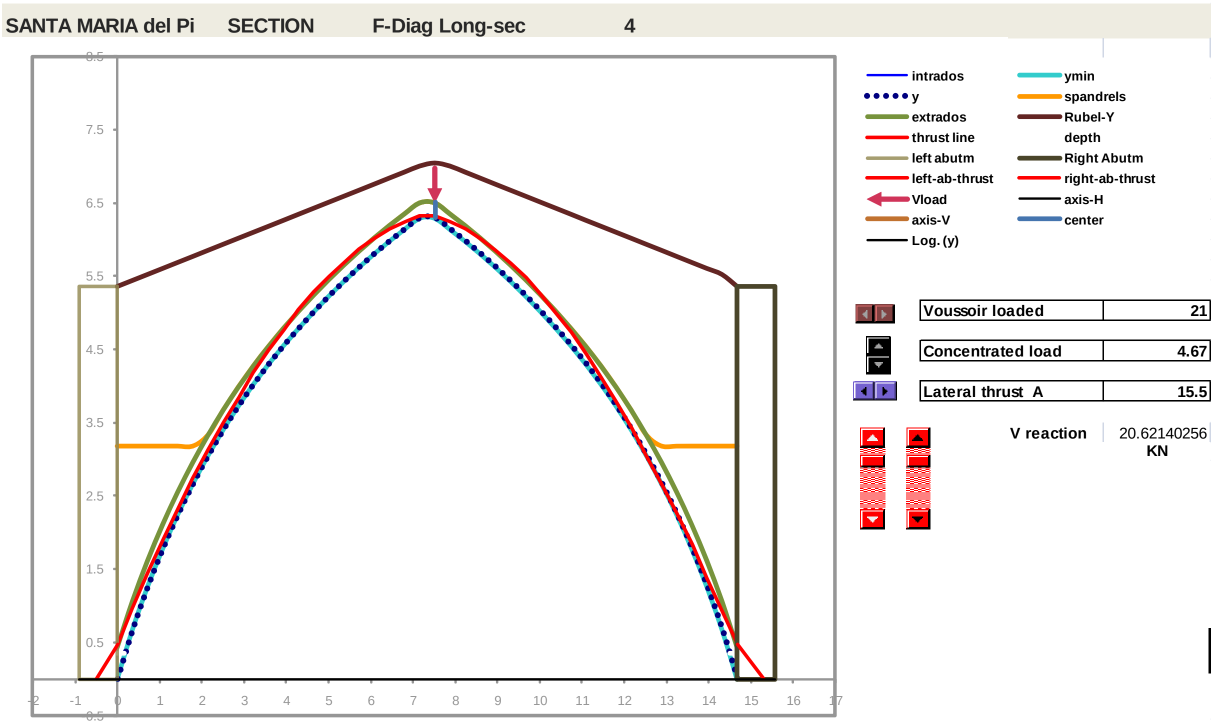
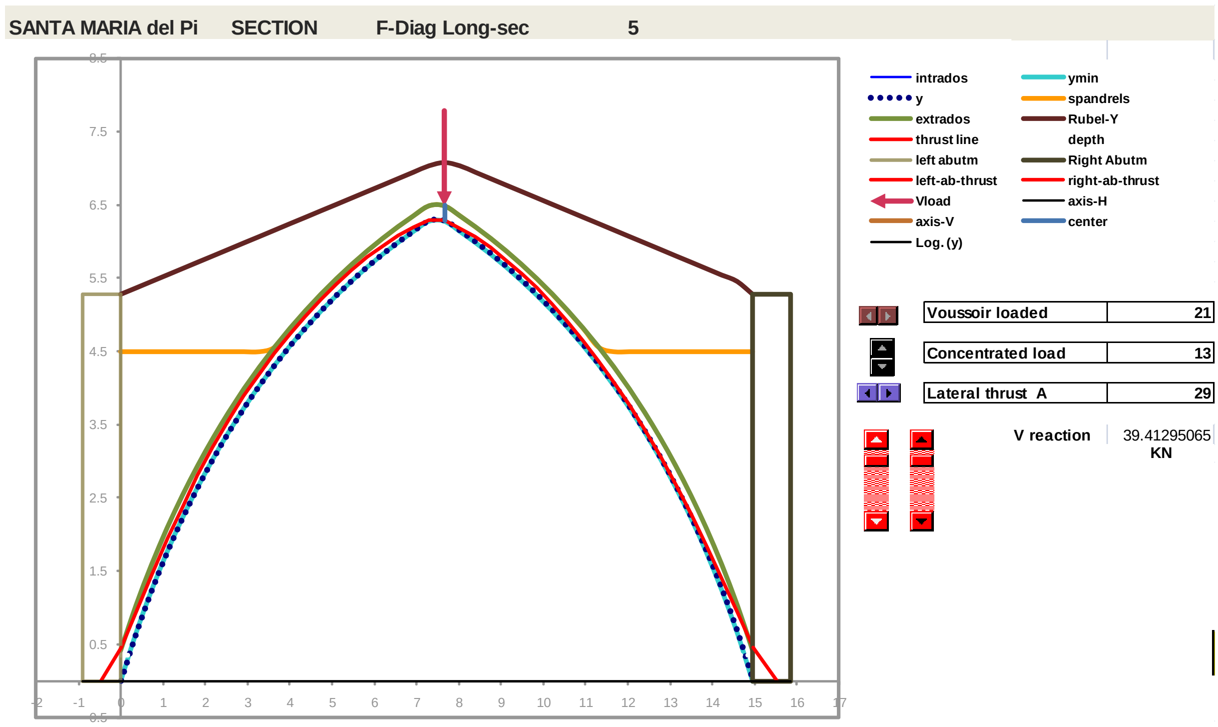
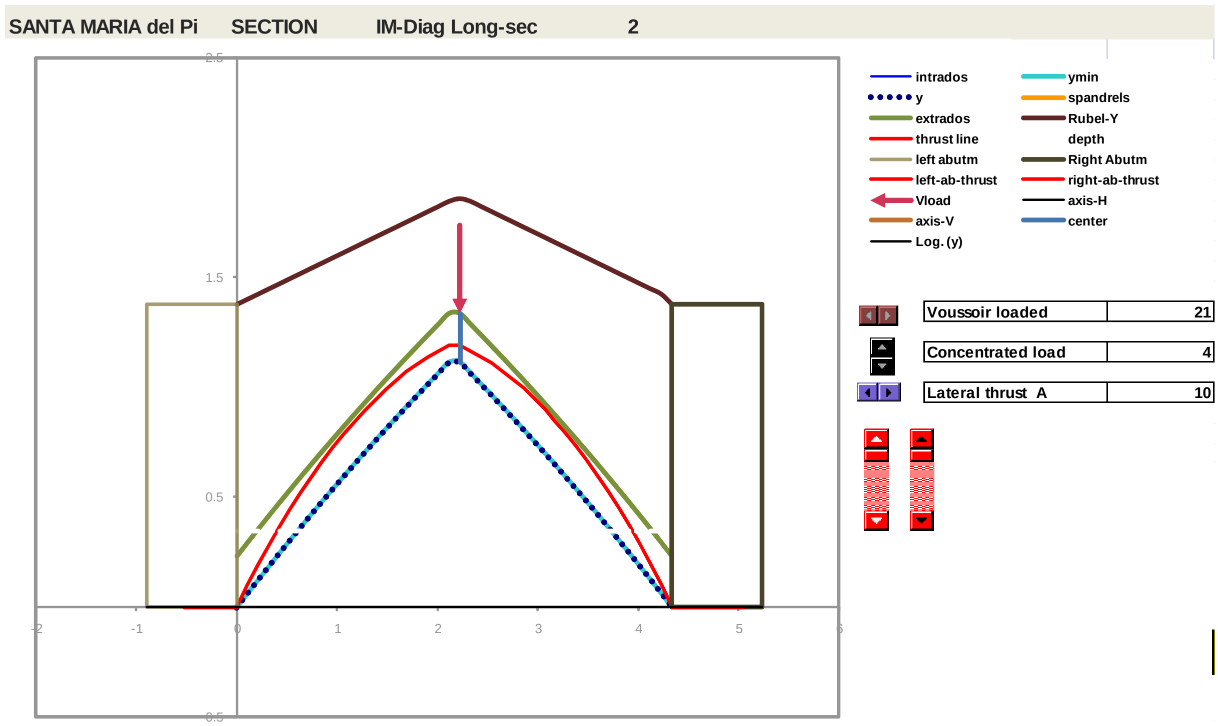
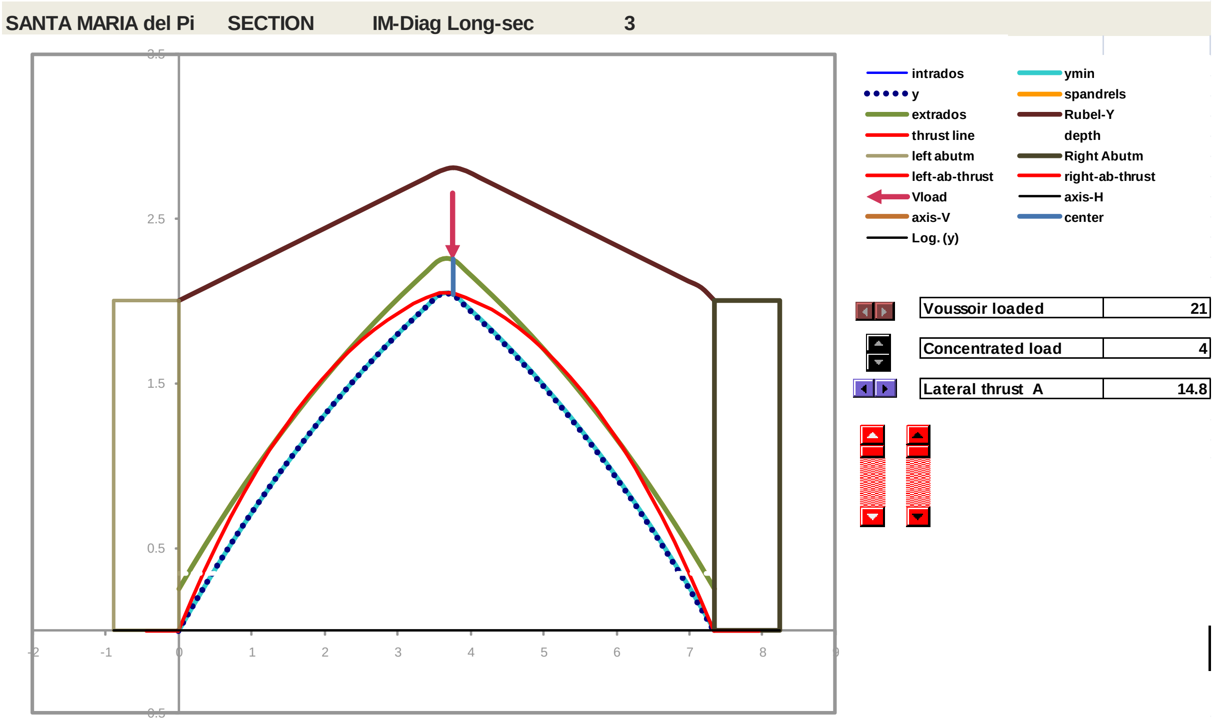
intrados
y
extrados
thrust line
left abutm
left-ab-thrust
Vload
axis-V
Log. (y)
ymin
spandrels
Rubel-Y
depth
Right Abutm
right-ab-thrust
axis-H
center
Voussoir loaded 21
Concentrated load 5
Lateral thrust A 15.7
V reaction 21.10839264 KN


SANTA MARIA del Pi SECTION F-Diag Long-sec 4
intrados
y
extrados
thrust line
left abutm
left-ab-thrust
Vload
axis-V
Log. (y)
ymin
spandrels
Rubel-Y
depth
Right Abutm
right-ab-thrust
axis-H
center
Voussoir loaded 21
Concentrated load 4.67
Lateral thrust A 15.5
V reaction 20.62140256 KN




## Santa Maria del Pi- Transversal Vault- New Diagonal sections

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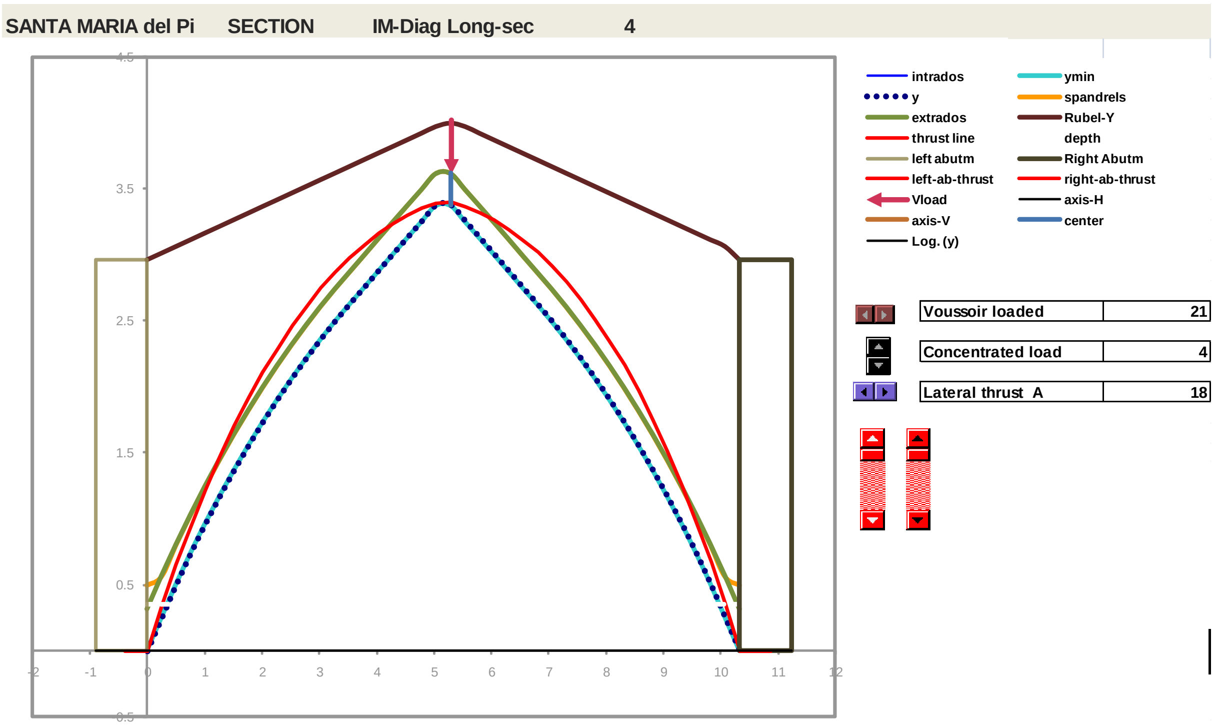
SECTION
IM-Diag Long-sec
3
intrados
y
extrados
thrust line
left abutm
left-ab-thrust
Vload
axis-V
Log. (y)
ymin
spandrels
Rubel-Y
depth
Right Abutm
right-ab-thrust
axis-H
center
Voussoir loaded 21
Concentrated load 4
Lateral thrust A 14.8
2.5
1.5
0.5
-1
0
1
2
3
4
5
6
7
8


SANTA MARIA del Pi
SECTION
IM-Diag Long-sec
4
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y
extrados
thrust line
left abutm
left-ab-thrust
Vload
axis-V
Log. (y)
ymin
spandrels
Rubel-Y
depth
Right Abutm
right-ab-thrust
axis-H
center
Voussoir loaded 21
Concentrated load 4
Lateral thrust A 18
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0.5
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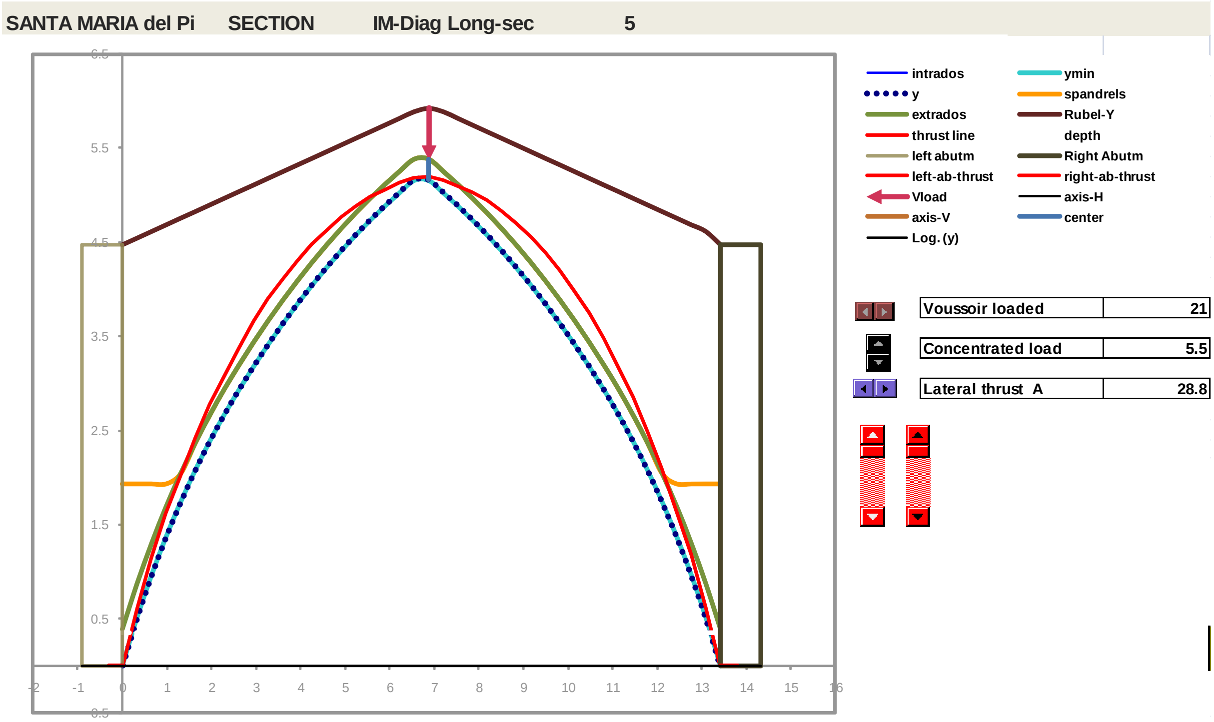
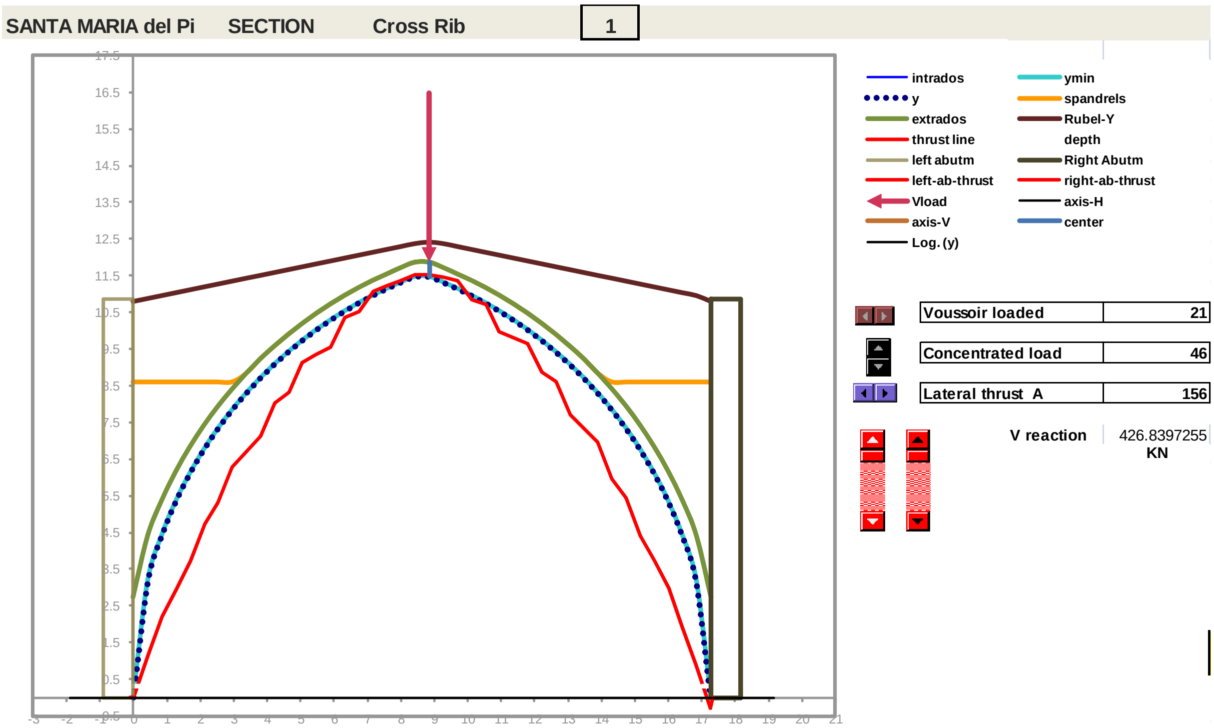
0
1
2
3
4
5
6
7
8
9
10
11

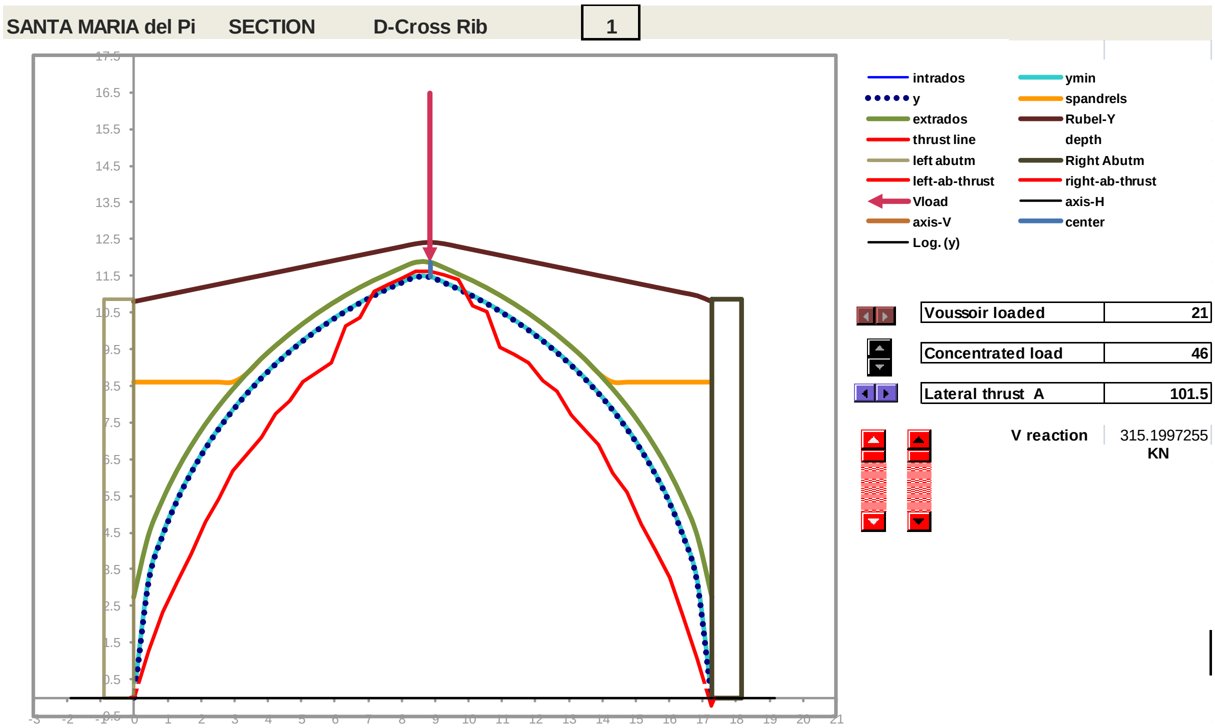



## Santa Maria del Pi- Cross Rib section – combined parallel

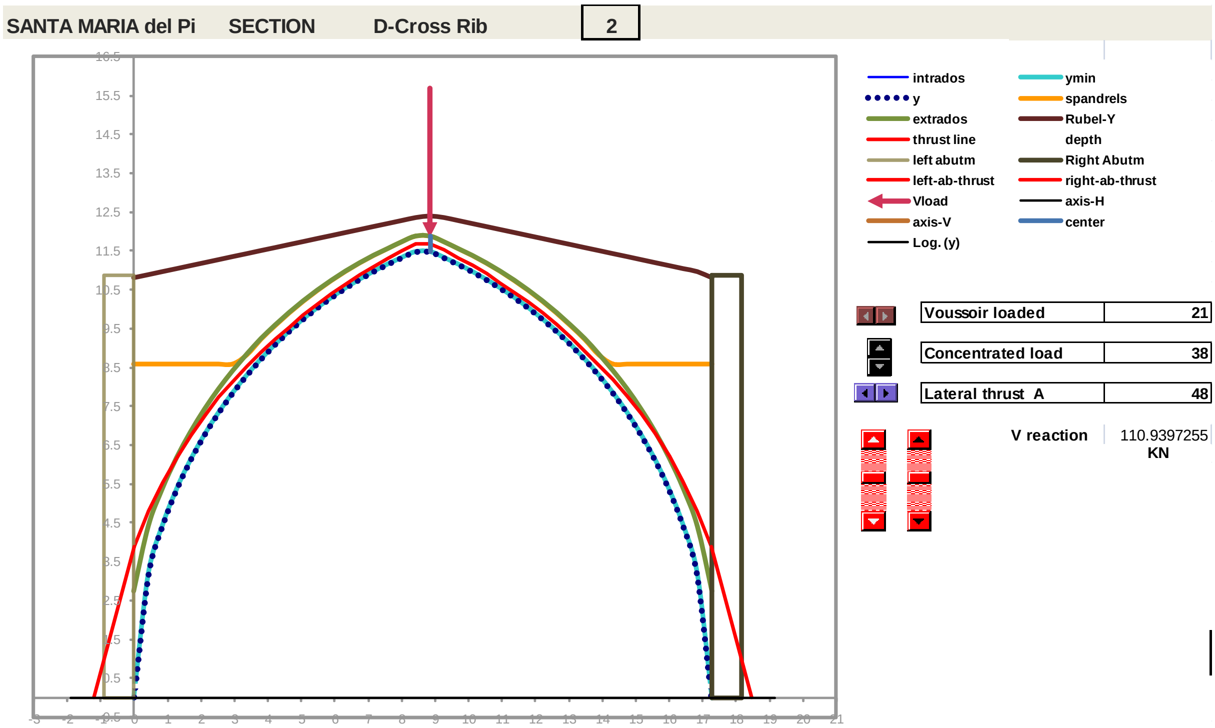


## Santa Maria del Pi- Cross Rib section – combined parallel & Diagonal

## By applying the loads of the webs:




## Without the load of the webs:



## Santa Maria del Mar- Diagonal sections:

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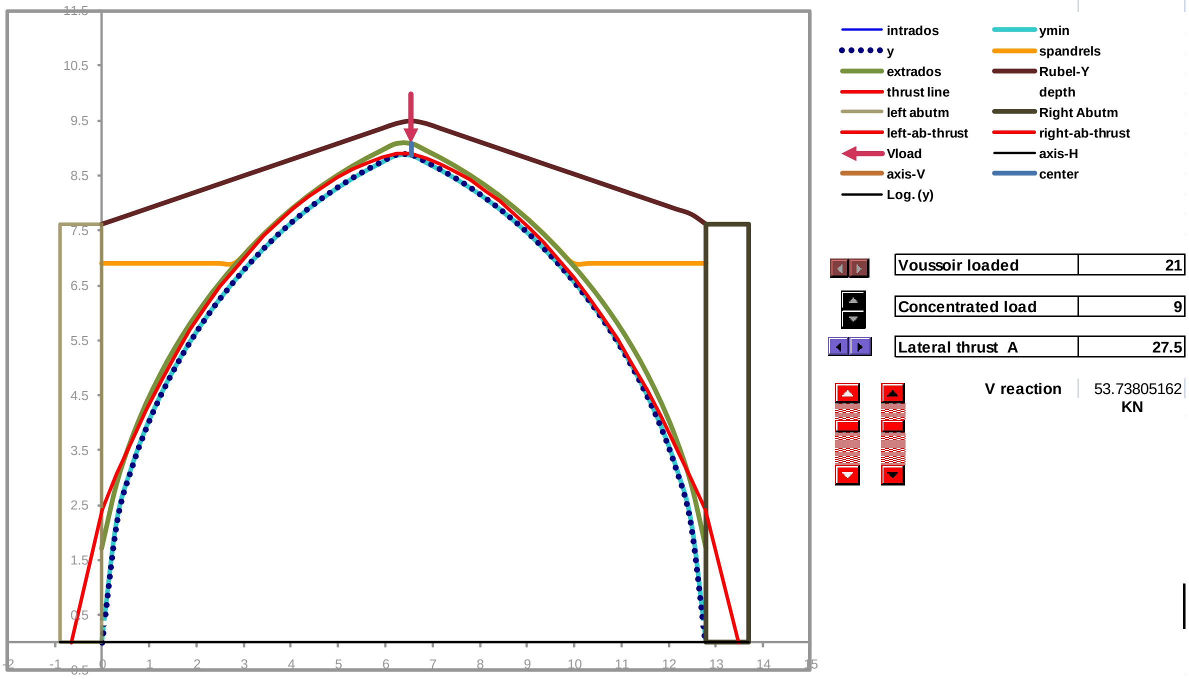
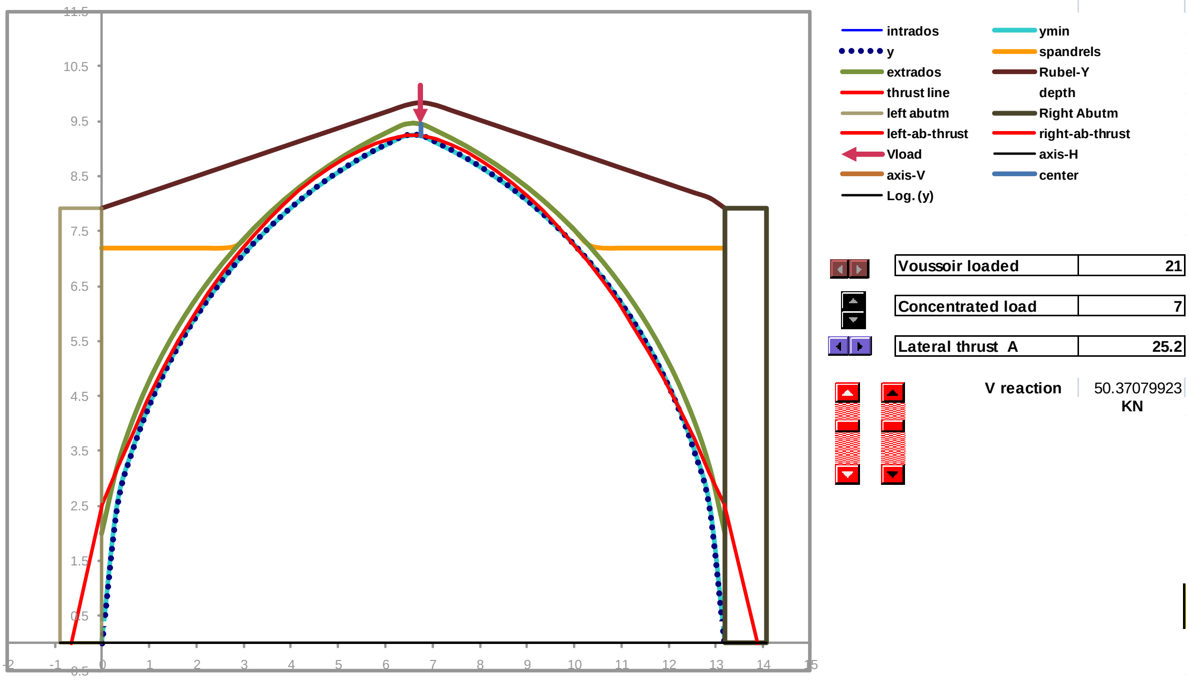
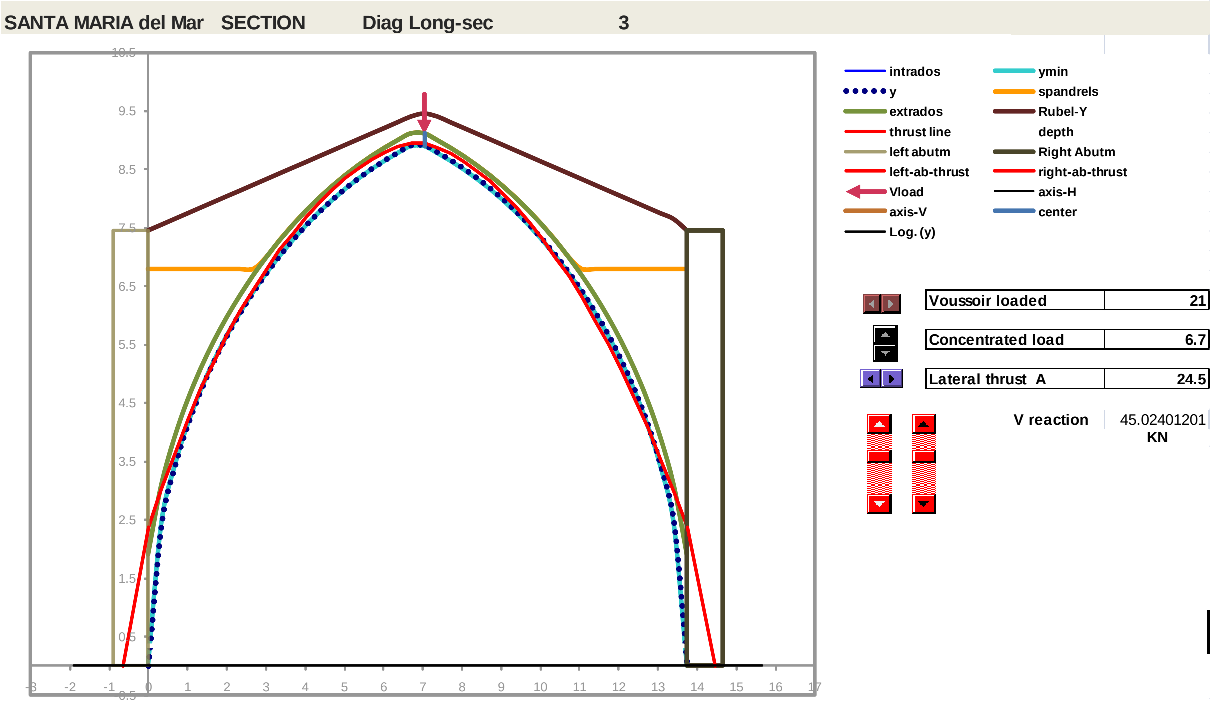
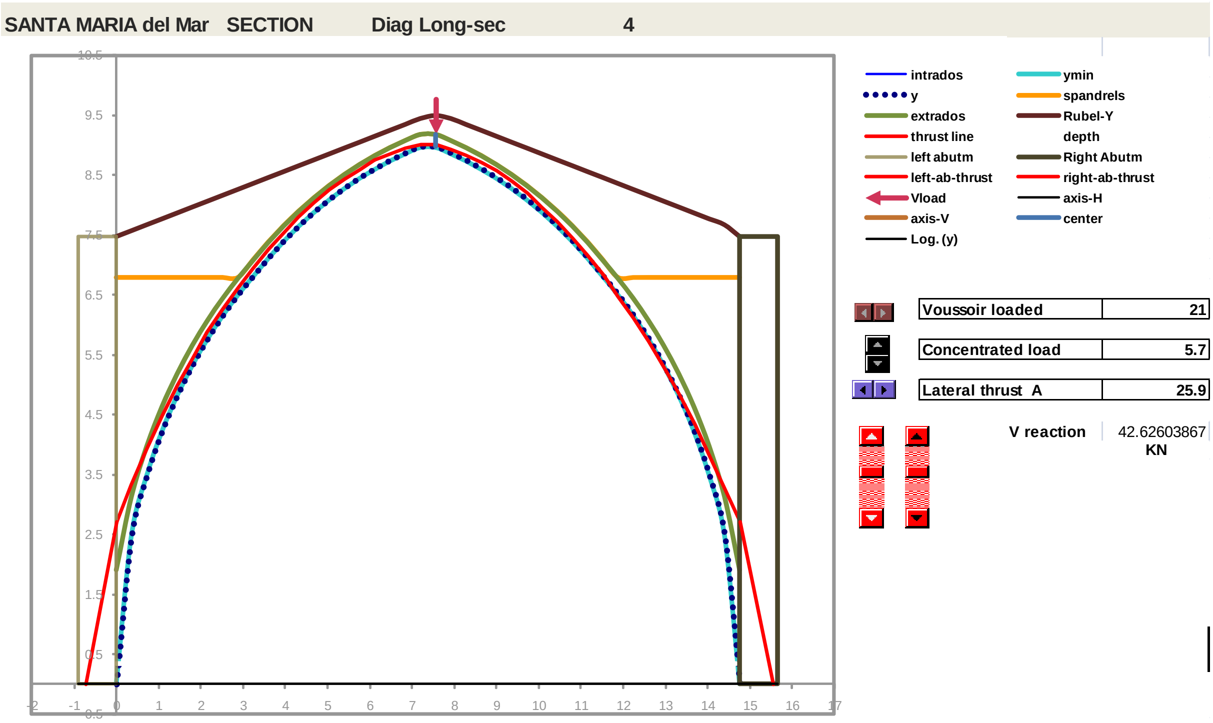



SANTA MARIA del Mar SECTION Diag Long-sec 2




SANTA MARIA del Mar SECTION Diag Long-sec 3
intrados
y
extrados
thrust line
left abutm
left-ab-thrust
Vload
axis-V
Log. (y)
ymin
spandrels
Rubel-Y
depth
Right Abutm
right-ab-thrust
axis-H
center
Voussoir loaded 21
Concentrated load 6.7
Lateral thrust A 24.5
V reaction 45.02401201
KN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extrados
thrust line
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left-ab-thrust
Vload
axis-V
Log. (y)
ymin
spandrels
Rubel-Y
depth
Right Abutm
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axis-H
center
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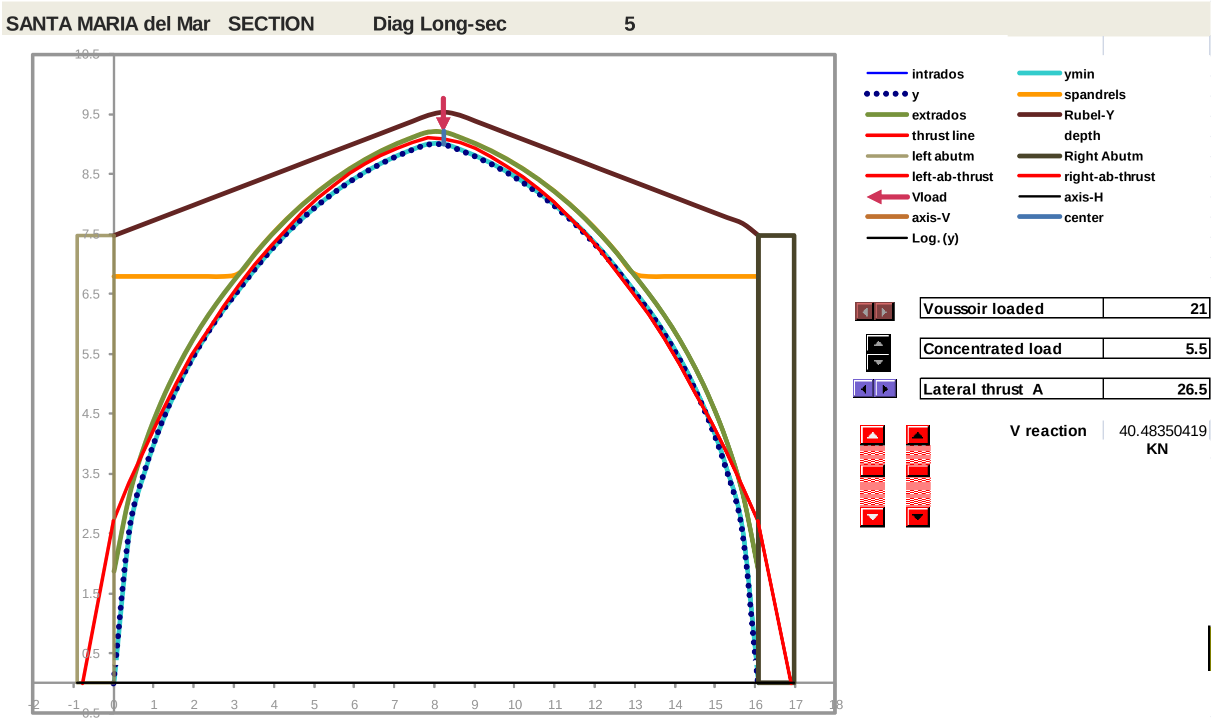
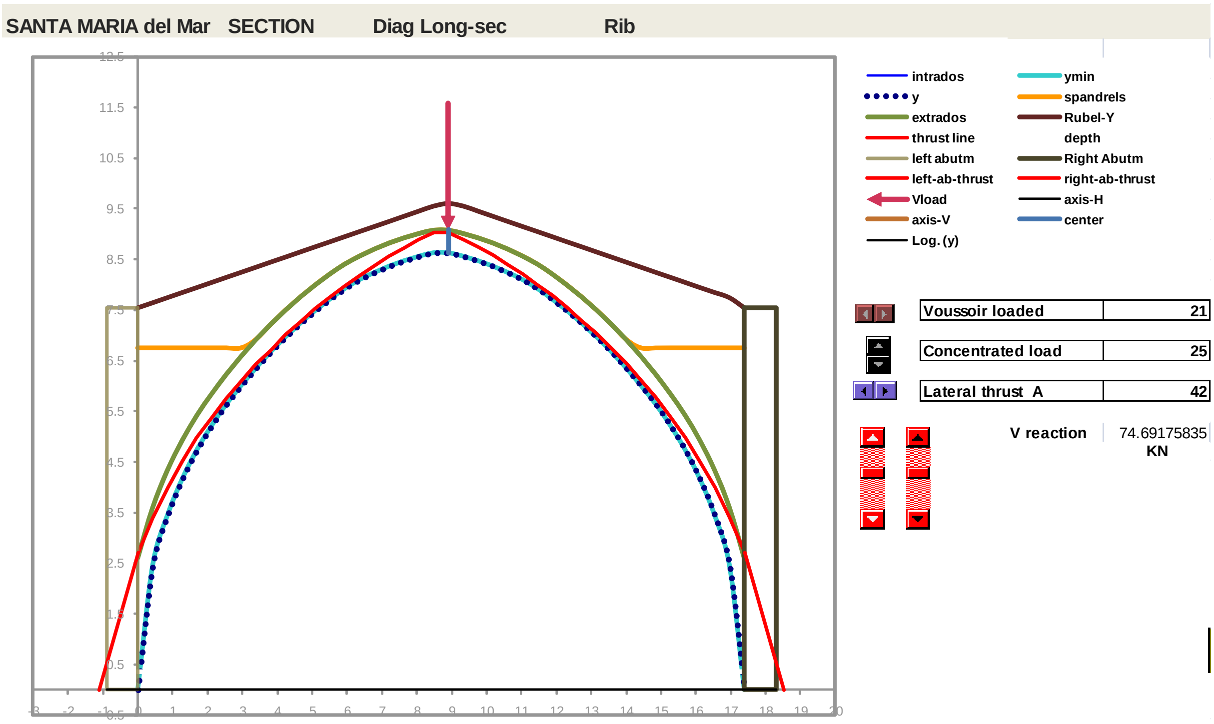
Concentrated load 5.7
Lateral thrust A 25.9
V reaction 42.62603867
KN




## Santa Maria del Mar- Cross Rib section

## By applying the loads of the infill:



UPC

ARCCHIP

## By applying the loads of the infill: